

Physics I: Mechanics and Relativity

One-Dimensional Kinematics and Calculus

Week 01

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Readings

1. **Feynman Lectures I, Ch. 8, §§8-2–8-3.** Read *Speed*, then *Speed as a derivative*. §8-2 introduces Achilles and the tortoise, and cuts a finite time into infinitely many pieces whose total stays finite. §8-3 takes the limit. These are the source for §§3–4 here; read them side by side. [feynmanlectures.caltech.edu/I_08.html](https://www.feynmanlectures.caltech.edu/I_08.html) ↗
2. **Kleppner & Kolenkow, §§1.7–1.9.** Position, displacement, velocity, acceleration, and the solution of the kinematical equations. Covers §§4–6 here.
3. **Kleppner & Kolenkow, §§2.7–2.8 and Notes 1.1, 1.5.** Units, dimensional analysis, approximation, and experimental uncertainty. Background for the units checks used throughout the course.
4. **Cole, §1.1.1.** Strategies for solving problems.

Overview

A ball is in the air. *How fast is it going right now?*

How can measurements over an interval describe motion at one instant? We start with position and displacement, then use limits to define velocity and acceleration. With these quantities defined, we solve for motion at constant velocity and constant acceleration.

Before calculating, sketch the motion and predict the result. Then compare the calculation with that prediction. If they disagree, identify which assumption or step failed. This is how we develop physical intuition beyond the motions already familiar from everyday experience.

You are not expected to memorize anything in this course.

How to use these notes

Each marker identifies a different kind of work: physical reasoning, derivation, prediction, or checking.

- ⚠ Common mistakes.
- ✍ **Think first:** a question to explore before the calculation.
- ✓ **Check:** test the result against intuition, limiting cases, signs, scale, or dimensions. A check of dimensional consistency is labeled ✓ **Units**.
- ∴ **Derivation:** follow the intermediate steps from the premise to the result.
- Σ **Useful math tool:** mathematical steps or notation used in the physics.
- 👁 **Watch:** assumptions and limits of a result.
- 📖 **Context:** historical or experimental background, excluded from exams.
- ↗ **Beyond the basics:** material past the course's scope, included for curiosity.
- 🎮 **Interactive animation:** a linked activity for testing a prediction.

1 Describing Motion in One Dimension

Kinematics describes motion; dynamics relates it to forces. We begin with motion along one spatial dimension.

To describe motion, specify the **position** $x(t)$ at each time. In one dimension, choose an origin, a positive direction, and a unit of length. These choices assign signed coordinates to locations; changing the coordinate labels does not change the motion.

✓ Units — Units and dimensions

Units specify a measurement scale: meters for length, seconds for time. Dimensions specify the kind of quantity: $[L]$ for length and $[T]$ for time. A length has the same dimension whether measured in meters or centimeters.

Both sides of an equation must have the same dimensions, as must quantities added together. For example, displacement divided by elapsed time has dimension $[L/T]$. This check can rule out an expression, but cannot prove it correct.

The **displacement** between two instants is $\Delta x = x(t_f) - x(t_i)$. Position depends on the chosen origin. Displacement does not change when the origin is shifted by a constant: the shift cancels in the difference. Its sign gives the direction of the net change; its magnitude need not equal the distance travelled.

A negative position means that the object is to the left of the chosen origin; it does not mean that the object is moving left.

Position as a function of time, $x(t)$, can be represented in several ways:

1. **Table of values.** A table lists the position at chosen times. It gives discrete snapshots and leaves the motion *between* them unspecified.
2. **Graph.** A plot of x against t shows the shape of the motion. The axis labels specify the quantities and units.

3. **Functional form.** An explicit formula for $x(t)$ gives the position at *every* instant, including instants we never measured. Its predictions depend on the assumptions of the model.

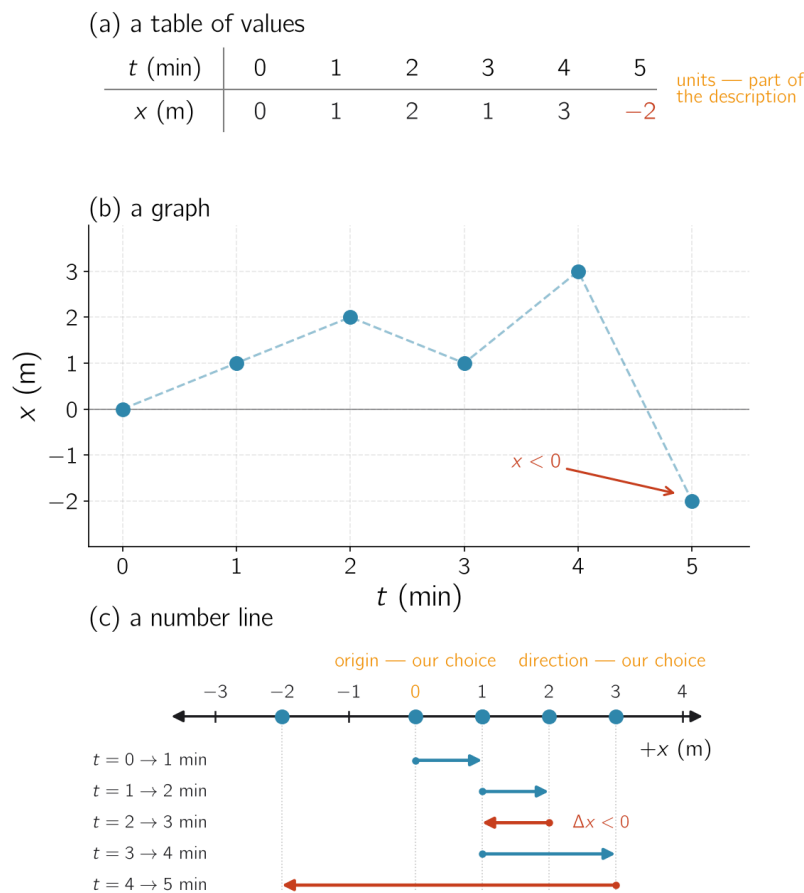


Figure 1: The same six measurements appear as a table (a), a graph (b), and a walk on a number line (c). The origin, the positive direction, and the units (x in meters, t in minutes) are all choices. Position is signed: the step from $t = 2$ to $t = 3$ min is the first with $\Delta x < 0$, and the walk ends at $x = -2$ m. The dashed line in (b) is only a guide.

Fitting arbitrary measured motion can require a complicated function. Ask first what stays constant: position, velocity, or acceleration. These three cases give different forms of $x(t)$:

Motion	Position $x(t)$
At rest	x_0
Constant velocity	$x_0 + vt$
Constant acceleration	$x_0 + v_0t + \frac{1}{2}at^2$

Here x_0 is the position at $t = 0$, v is a constant velocity, v_0 is the initial velocity, and a is a constant acceleration. At this stage, the distinction is the shape of $x(t)$: constant, linear, or quadratic.

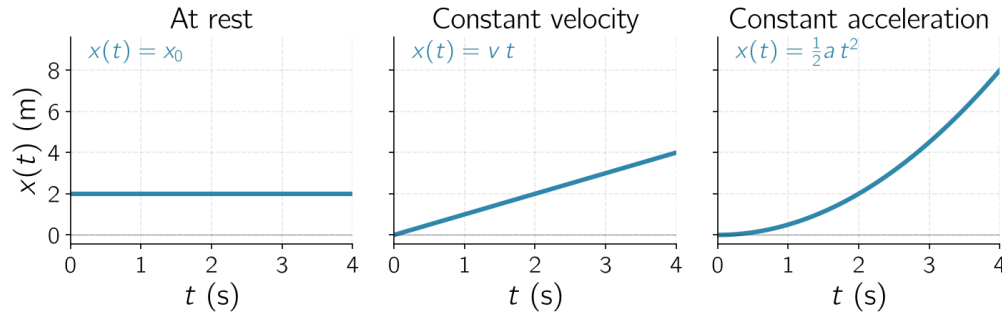


Figure 2: Constant, linear, and quadratic position functions. The two moving examples use $x_0 = 0$; the quadratic example also uses $v_0 = 0$.

2 Average Velocity

The **average velocity** over a time interval $[t_i, t_f]$ is the displacement divided by the elapsed time:

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}.$$

Its dimensions are $[L/T]$; its SI unit is m/s. The bar in $\bar{v}$ means an average over an interval; it is not a vector arrow. On an x - t graph, $\bar{v}$ is the slope of the secant through the two endpoints. Those endpoints do not tell us how the object moved in between: different journeys can have the same displacement and elapsed time.

For **constant-velocity motion** $x(t) = x_0 + vt$, every interval has the same average velocity v .

Now take $x(t) = \frac{1}{2}at^2$, with constant acceleration $a > 0$, starting from rest at the origin. At $t = 1$ s and 2 s, the positions are $\frac{1}{2}a(1\text{ s})^2$ and $2a(1\text{ s})^2$. Over intervals starting at $t = 0$:

- From 0 to 1 s: $\bar{v} = \frac{\frac{1}{2}a(1\text{ s})^2 - 0}{1\text{ s} - 0} = \frac{1}{2}a \cdot (1\text{ s})$.
- From 0 to 2 s: $\bar{v} = \frac{\frac{1}{2}a(2\text{ s})^2 - 0}{2\text{ s} - 0} = a \cdot (1\text{ s})$.
- From 0 to 3 s: $\bar{v} = \frac{3}{2}a \cdot (1\text{ s})$.

For any interval with $t_f > t_i$,

$$\bar{v} = \frac{\frac{1}{2}at_f^2 - \frac{1}{2}at_i^2}{t_f - t_i} = \frac{a}{2}(t_f + t_i).$$

From 0 to T , this gives $\bar{v} = aT/2$. The average describes the chosen interval but does not determine the instantaneous velocity at a particular time.

⚠ Average velocity vs. average speed

In one dimension velocity is signed; its sign is the direction. Walk 3 m east and then 3 m west in 10 s: your displacement is zero, so your average velocity is zero, but you covered 6 m, so your average speed is 0.6 m/s. Average speed is total distance over time, not the magnitude of the average velocity, and it is never smaller than $|\bar{v}|$.

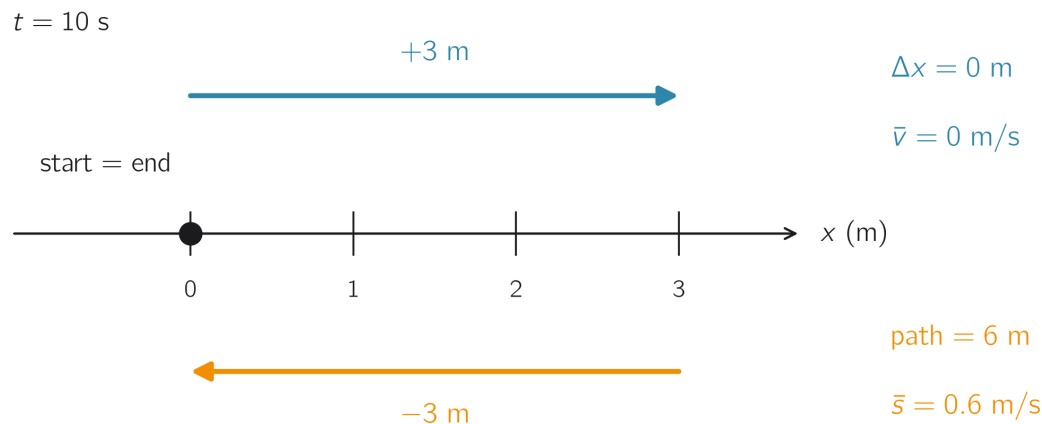


Figure 3: A walker covers 3 m east and then 3 m west in 10 s. The two legs cancel, so the displacement vanishes and the average velocity $\bar{v}$ is zero. The path length is 6 m, so the average speed is 0.6 m/s.

3 Zeno's Paradox and Infinite Series

For constant velocity, the same slope describes every time interval. When velocity changes, an interval average does not by itself define the velocity at an instant. Algebra gives us the averages; the new step is to understand their limit as the interval shrinks. This leads to differential calculus. First, consider Zeno's question: must infinitely many stages of motion take infinitely long?

Setup. Achilles, the fastest runner in Greece, races a tortoise. The tortoise gets a head start of $x_0 > 0$. Achilles runs with constant velocity v_A ; the tortoise crawls with constant velocity $0 \leq v_T < v_A$.

Zeno's argument. Each time Achilles reaches the tortoise's previous position, the tortoise has moved farther ahead. This repeats through infinitely many steps. Zeno therefore concludes that Achilles never catches the tortoise.

✓ Units — Dimensions and a limiting case

The answer must have dimensions of time, $[T]$. We have one length, x_0 $[L]$, and two velocities v_A, v_T $[L/T]$. Dividing a length by a velocity gives a time, but dimensions alone do not determine which velocity or combination belongs in the denominator. The

physical denominator is the rate at which Achilles closes the gap, $v_A - v_T$. Check the limit $v_A = v_T$: the closing rate vanishes, so Achilles never catches up and $t^* \rightarrow \infty$. We predict

$$t^* \sim \frac{x_0}{v_A - v_T}$$

The dimensions are $[L]/[L/T] = [T]$. The calculation below tests the expression; dimensional consistency alone is not enough.

3.1 Resolution via geometric series

Σ The geometric series

For $|r| < 1$, successive powers of r tend to zero. Start with the finite sum

$$S_N = 1 + r + r^2 + \cdots + r^N.$$

Multiply by r and subtract:

$$S_N - rS_N = 1 - r^{N+1} \quad \Rightarrow \quad S_N = \frac{1 - r^{N+1}}{1 - r}.$$

As $N \rightarrow \infty$, the term r^{N+1} goes to zero, so

$$\boxed{\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r}}, \quad |r| < 1.$$

The condition matters. The sum converges for $|r| < 1$. For $|r| \geq 1$, the terms do not tend to zero, so the series diverges.

Let us follow Zeno's logic and add up the infinitely many step-times. Define $r \equiv v_T/v_A < 1$. This dimensionless number is the common ratio of successive step durations. It is shorthand for the velocity ratio, not an additional parameter.

- **Step 0.** Achilles covers the head start x_0 in time $\Delta t_0 = x_0/v_A$. The tortoise advances by $v_T \Delta t_0 = r x_0$. This first interval ends when Achilles reaches the tortoise's initial position, not when he catches the moving tortoise. That is why its duration is x_0/v_A , rather than $x_0/(v_A - v_T)$.
- **Step 1.** Achilles covers $r x_0$ in time $\Delta t_1 = r x_0/v_A = r \cdot \Delta t_0$. The tortoise advances by $r^2 x_0$.
- **Step k .** $\Delta t_k = r^k \Delta t_0$.

The total time is the geometric series

$$t^* = \sum_{k=0}^{\infty} \Delta t_k = \Delta t_0 \sum_{k=0}^{\infty} r^k = \frac{\Delta t_0}{1 - r}.$$

Plugging in $\Delta t_0 = x_0/v_A$ and $r = v_T/v_A$,

$$t^* = \frac{x_0}{v_A - v_T}.$$

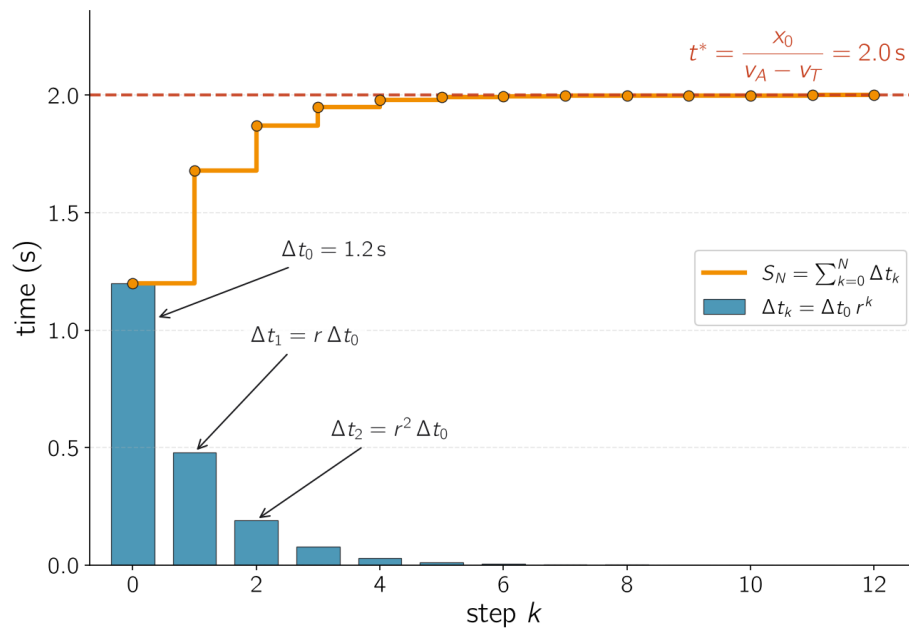


Figure 4: Zeno's infinite sum converges. Each blue bar is the duration $\Delta t_k = \Delta t_0 r^k$ of one step, with $\Delta t_0 = 1.2$ s and $r = v_T/v_A = 0.4$; the orange staircase is the running total $S_N = \sum_{k=0}^N \Delta t_k$. It flattens onto the asymptote $t^* = x_0/(v_A - v_T) = 2.0$ s. Infinitely many steps take a finite total time.

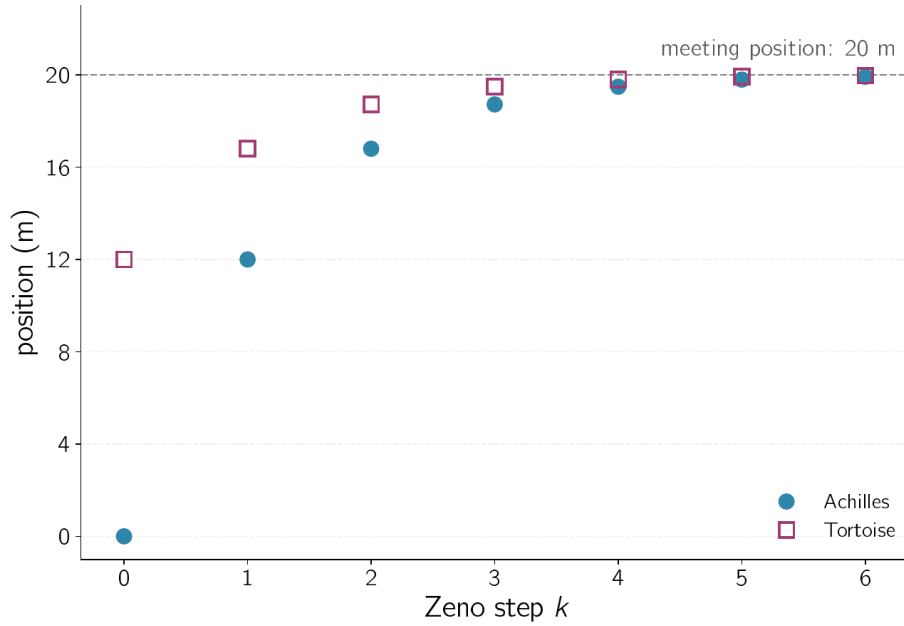


Figure 5: Positions at the start of step k , with $x_0 = 12$ m, $v_A = 10$ m/s, and $v_T = 4$ m/s. Achilles reaches the tortoise's preceding position at each step. The gap $x_0 r^k$ is positive at every finite step, although the markers eventually overlap at this scale. Both sequences approach 20 m. Equal steps on this axis are not equal time intervals.

3.2 Algebraic catch-up solution

For the assumed constant velocities, we can find the meeting point directly by equating positions. Let $x = 0$ be Achilles' start, with the tortoise starting at x_0 . Then

$$x_A(t) = v_A t, \quad x_T(t) = x_0 + v_T t.$$

They meet when $x_A(t^*) = x_T(t^*)$:

$$v_A t^* = x_0 + v_T t^* \implies t^* = \frac{x_0}{v_A - v_T}. \quad \checkmark$$

The meeting position, measured from Achilles' starting point, is

$$\boxed{x^* = v_A t^* = \frac{v_A x_0}{v_A - v_T} = \frac{x_0}{1 - r}} = x_0 + v_T t^*.$$

For the plotted example, $t^* = 2$ s and $x^* = 20$ m: Achilles travels 20 m and the tortoise 8 m beyond its 12 m head start. This is the common position limit in the step plot.

Both methods give the same meeting time and position. Equating positions is algebraic; summing the steps answers Zeno's objection about infinitely many stages.

✓ Checks

- Units: $[L]/[L/T] = [T]$. ✓
- $v_A \rightarrow v_T$ from above: $t^* \rightarrow \infty$. Achilles never catches up. ✓
- $v_T \rightarrow 0$ (stationary tortoise): $t^* \rightarrow x_0/v_A$. Achilles just covers the head start. ✓
- $x_0 \rightarrow 0$: $t^* \rightarrow 0$. They start together. ✓

3.3 Why we need calculus

Neither catch-up calculation requires differential calculus: we already know the position functions. But when velocity changes, how do we describe motion at an instant, or predict a trajectory we do not yet know? Calculus gives us a direct way to connect position, velocity, and acceleration through their rates of change. We begin by defining instantaneous velocity.

4 Instantaneous Velocity and the Derivative

When velocity changes, the average velocity depends on the interval chosen. To find the velocity *right now*, make the interval shorter and determine the value that $\bar{v}$ approaches.

Definition. The **instantaneous velocity** at time t is

$$v(t) = \lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{dx}{dt}.$$

The notation dx/dt denotes the derivative of position with respect to time. We also write $v(t) = \dot{x}(t)$, read as x -dot. A dot always means a derivative with respect to time.

Geometrically, $v(t)$ is the slope of the **tangent** to the x - t curve at time t . Secant lines approach that tangent as the interval shrinks.

Σ Differentials

How much does the position change during a little bit of time? Velocity tells us the rate: at time t_0 , a small time step Δt gives a displacement approximately equal to $v(t_0)\Delta t$. For constant velocity, twice the time gives twice the displacement.

We write this local relation using **differentials**:

$$dx = v(t_0) dt.$$

Here dt is the differential of time, a chosen time increment, and dx is the corresponding differential of position. The units agree, since $(\text{m/s})\text{s} = \text{m}$.

More precisely, dx is the displacement along the tangent line. For $dt = \Delta t$, the actual displacement is

$$\Delta x = x(t_0 + \Delta t) - x(t_0) \approx dx = v(t_0)\Delta t.$$

For constant velocity this equality is exact. For changing velocity it becomes a better local approximation as the interval shrinks.

This also suggests how to recover position: add the small displacements over successive time steps.

Further reading: Kleppner and Kolenkow, second edition, Note 1.4, Differentials.

Zeno's second paradox: the arrow. At each instant the arrow occupies a definite position. Zeno takes this to mean that it is at rest, and concludes that motion is impossible. But a single snapshot does not determine velocity. Setting $\Delta t = 0$ gives $\Delta x = 0$ and the undefined expression $0/0$. Instead, can the ratio $\Delta x/\Delta t$ approach a finite value as a nonzero interval shrinks to zero? That limit, when it exists, is the instantaneous velocity. Every quotient used in the limit has a nonzero time interval; we never divide by zero. We ask what value those quotients approach.

In the **secant-to-tangent activity** ↗, vary the time interval and compare the average velocity with the tangent slope.

△ Velocity as a function

“Velocity” can mean a function of time, $v(t)$, or a single number at a particular instant. When the meaning could be ambiguous, write the time-dependence explicitly: $v(t)$, not v . x may stand for $x(t)$; x_0 or $x(t_1)$ denotes a particular value.

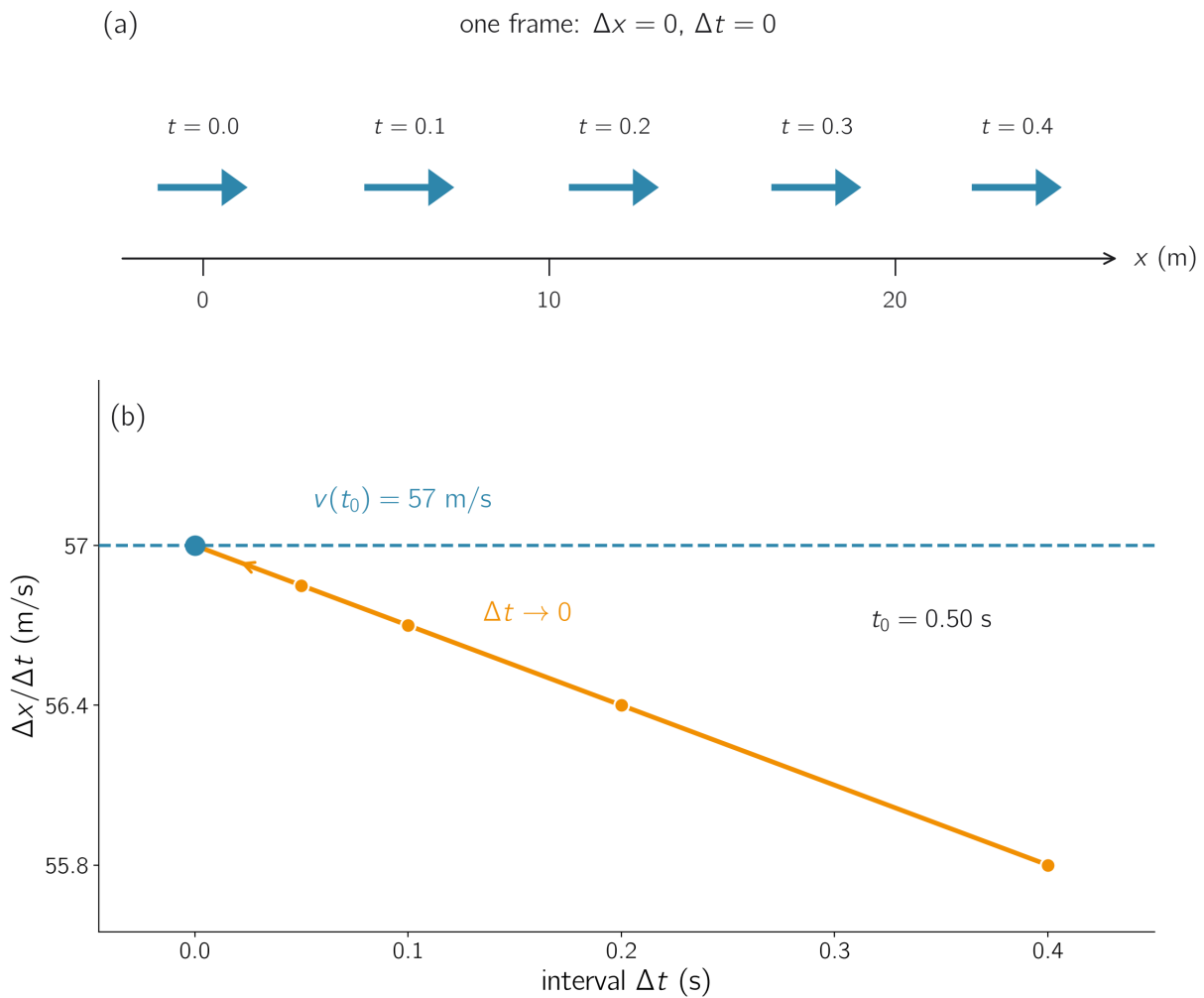


Figure 6: An arrow is launched at 60 m/s with constant deceleration of 6 m/s^2 . (a) A single strobe frame shows one position; comparing frames shows motion. (b) At $t_0 = 0.50 \text{ s}$, the average velocity over $[t_0, t_0 + \Delta t]$ rises to the finite limit $v(t_0) = 57 \text{ m/s}$ as $\Delta t \rightarrow 0$.

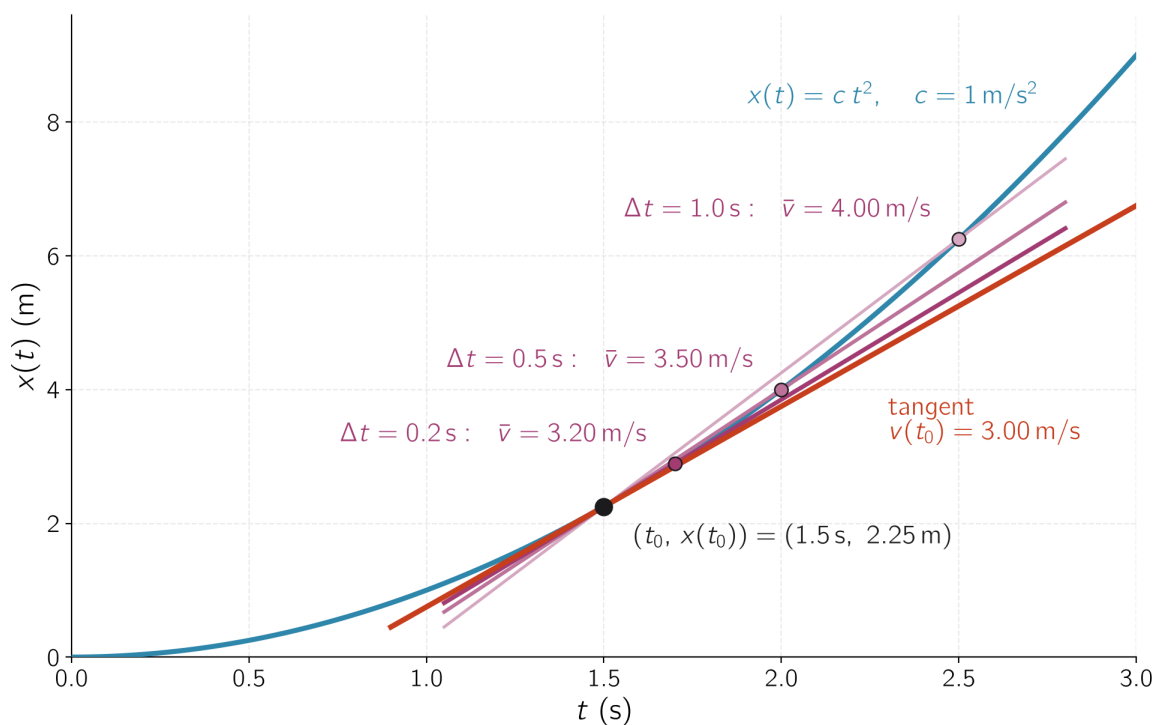


Figure 7: Secant slopes approach the instantaneous velocity. The blue curve is $x(t) = \frac{1}{2}at^2$ with $a = 2 \text{ m/s}^2$, and the black dot is $(t_0, x(t_0)) = (1.5 \text{ s}, 2.25 \text{ m})$. Each magenta secant passes through that point and a second point on the curve $\Delta t = 1.0, 0.5$ or 0.2 s later, with slopes $\bar{v} = 4.00, 3.50$ and 3.20 m/s ; the lighter the line, the wider the interval. As $\Delta t \rightarrow 0$ the secants rotate onto the red tangent, whose slope is $v(t_0) = 3.00 \text{ m/s}$.

4.1 Motion at Constant Velocity

A **differential equation** relates a function to its derivatives. For motion at constant velocity v_0 , it takes the form

$$\dot{x}(t) = v_0, \quad v_0 = \text{constant.}$$

Let us guess a solution. A constant slope suggests a straight line: try $x(t) = v_0 t$. Differentiating gives $\dot{x} = v_0$, so the guess satisfies the equation. But it also assumes $x(0) = 0$. Adding any constant C leaves the derivative unchanged, so $x(t) = v_0 t + C$ is a solution too.

The equation determines the displacement, $x(t) - x(0) = v_0 t$, but does not tell us where the object started. We supply that information through the initial condition $x(0) = x_0$, which fixes $C = x_0$:

$$x(t) = x_0 + v_0 t.$$

The numerical value of x_0 depends on the chosen spatial origin. Shifting that origin changes every position by the same constant, leaving the displacement and velocity unchanged. Once an origin is chosen, the initial position is the additional datum needed to select one trajectory.

∴ Recovering position from velocity

An integral is a sum of small contributions. Here we add displacements to recover position. Write the velocity explicitly as a function of time:

$$\frac{dx}{dt} = v(t), \quad dx = v(t) dt.$$

To add the displacement from t_0 to $t > t_0$, divide the interval into N pieces, with $\Delta t = (t - t_0)/N$ and $t_k = t_0 + k\Delta t$. In each short piece, $\Delta x_k \approx v(t_k)\Delta t$. For continuous velocity, taking the limit gives

$$x(t) - x(t_0) = \lim_{N \rightarrow \infty} \sum_{k=0}^{N-1} v(t_k) \Delta t = \int_{t_0}^t v(t') dt'.$$

The variable t' labels the intermediate times; t is the final time. Adding the initial position yields

$$x(t) = x(t_0) + \int_{t_0}^t v(t') dt'.$$

This is the meaning of integrating $dx = v(t) dt$. The integral gives signed displacement, not total distance: negative velocity contributes negatively. For constant velocity, it gives $x(t) = x(t_0) + v_0(t - t_0)$, the same solution found by guessing and checking.

Σ Basic derivatives

Here C is constant, n is a positive integer, and $u = u(t)$ and $w = w(t)$ are differentiable functions.

$f(t)$	df/dt	$f(t)$	df/dt
C	0	t	1
t^n	nt^{n-1}	$\sin(\omega t)$	$\omega \cos(\omega t)$
$\cos(\omega t)$	$-\omega \sin(\omega t)$	Cu	$C\dot{u}$
$e^{\lambda t}$	$\lambda e^{\lambda t}$		

The constants ω and λ have dimensions of inverse time, so ωt and λt are dimensionless; angles are in radians. For a dimensionless variable z , $d(\sin z)/dz = \cos z$ and $d(\cos z)/dz = -\sin z$.

$$\frac{d}{dt}(u + w) = \dot{u} + \dot{w}, \quad \frac{d}{dt}(uw) = \dot{u}w + u\dot{w}.$$

For the product rule, write $\Delta(uw) = w\Delta u + u\Delta w + \Delta u\Delta w$, divide by Δt , and take the limit. The last term vanishes: $(\Delta u/\Delta t)\Delta w \rightarrow \dot{u}0 = 0$.

Return to Achilles and the tortoise. We know their velocities at every time, $\dot{x}_A = v_A$ and $\dot{x}_T = v_T$, and their initial positions, $x_A(0) = 0$ and $x_T(0) = x_0$. Guess a straight line for each:

$$x_A(t) = c_A t + b_A, \quad x_T(t) = c_T t + b_T.$$

Differentiating and imposing the equations of motion fixes the slopes, $c_A = v_A$ and $c_T = v_T$. Evaluating at $t = 0$ fixes the intercepts, $b_A = 0$ and $b_T = x_0$. Thus

$$x_A(t) = v_A t, \quad x_T(t) = x_0 + v_T t.$$

The velocity in each equation is already specified; only the additive constant remains to be fixed by an initial condition.

On an x - t graph, draw Achilles' line from the origin with slope v_A , and the tortoise's line from height x_0 with slope v_T . Their intersection gives the meeting time and position. For $x_0 > 0$ and $v_A > v_T$, the same construction is expressed algebraically as

$$v_A t^* = x_0 + v_T t^*, \quad t^* = \frac{x_0}{v_A - v_T}, \quad x^* = \frac{v_A x_0}{v_A - v_T}.$$

For our example, the intersection is $(t^*, x^*) = (2 \text{ s}, 20 \text{ m})$. We have recovered the functions assumed in the earlier catch-up calculation from equations of motion and initial conditions. Finding their intersection is still algebra.

∴ Shifting the Space and Time Origins

Choose a new spatial origin displaced by a constant ℓ , and start a new clock a constant τ later. Keep the same units and positive direction. The same event then has coordinates

$$x' = x - \ell, \quad t' = t - \tau.$$

Both differences are unchanged: $\Delta x' = \Delta x$ and $\Delta t' = \Delta t$. Every average velocity is therefore unchanged, and so is its instantaneous limit:

$$v' = \lim_{\Delta t' \rightarrow 0} \frac{\Delta x'}{\Delta t'} = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = v.$$

For $x(t) = x_0 + v_0 t$, substituting $t = t' + \tau$ gives $x'(t') = (x_0 + v_0 \tau - \ell) + v_0 t'$. The intercept changes, but the slope does not. These are different coordinate descriptions of the same motion; the comparison is always at the same physical event.

✍ Crossing position graphs

Two objects move along a line. On the same x - t axes, A has a steep straight graph starting low; B has a shallower straight graph starting high. Both slopes are positive. The lines cross at $t = 2$ s. Which object is moving faster at the crossing: A, B, or neither? What does the crossing tell you?

Compare your prediction with the **graph activity** ↗.

⚠ Position and slope

On an x - t graph, height gives position; slope gives velocity. A crossing means equal positions, not equal velocities. Speed is the magnitude of the slope.

A secant slope is $\Delta x / \Delta t$, not generally x / t . The latter is a slope measured from the coordinate origin, which need not lie on the curve.

Comparison. At the crossing, both objects have the same position. A's positive slope is larger, so A is moving faster before, at, and after the crossing.

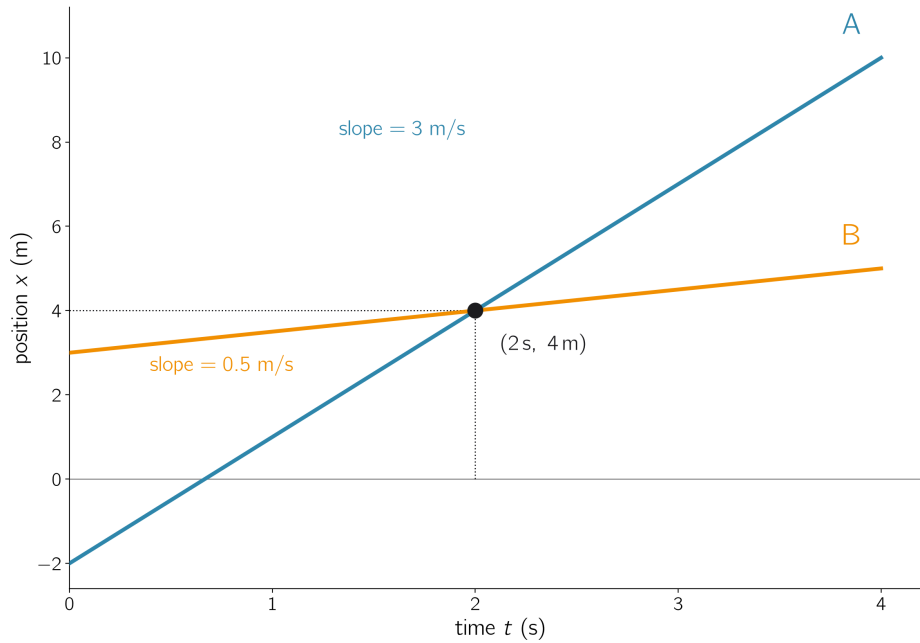


Figure 8: Object A follows $x_A = -2 + 3t$ and object B follows $x_B = 3 + 0.5t$, in meters and seconds. The two lines meet at $t = 2\text{ s}$, $x = 4\text{ m}$, so both objects are at the same place. Their slopes differ by a factor of six, so A moves six times faster there.

4.2 Worked example: a quadratic position function

Consider the position function $x(t) = \frac{1}{2}at^2$, with constant a and $x(0) = 0$. We now find its instantaneous velocity from the definition:

$$\begin{aligned} v(t) &= \lim_{\Delta t \rightarrow 0} \frac{\frac{1}{2}a(t + \Delta t)^2 - \frac{1}{2}at^2}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{at \Delta t + \frac{1}{2}a(\Delta t)^2}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \left(at + \frac{1}{2}a \Delta t \right) \\ &= at. \end{aligned}$$

So the velocity at time t is $v(t) = at$. For comparison: at $t = 1\text{ s}$, $v = a(1\text{ s})$, while over $[0, 1\text{ s}]$ we found $\bar{v} = \frac{1}{2}a(1\text{ s})$. For motion that starts at rest and speeds up linearly, the velocity at the end of an interval beginning at release is *twice* the average over it.

For $x = \frac{1}{2}at^2$, the gap between the forward-interval average velocity and $v(t) = at$ is $\frac{1}{2}a \Delta t$.

Adding $x(t_0)$ to the small displacement gives a local approximation for position:

$$x(t_0 + \Delta t) \approx x(t_0) + v(t_0) \Delta t.$$

This is the **first-order Taylor approximation** at t_0 . The figure shows how the smooth trajectory approaches this straight line when we zoom in near that time.

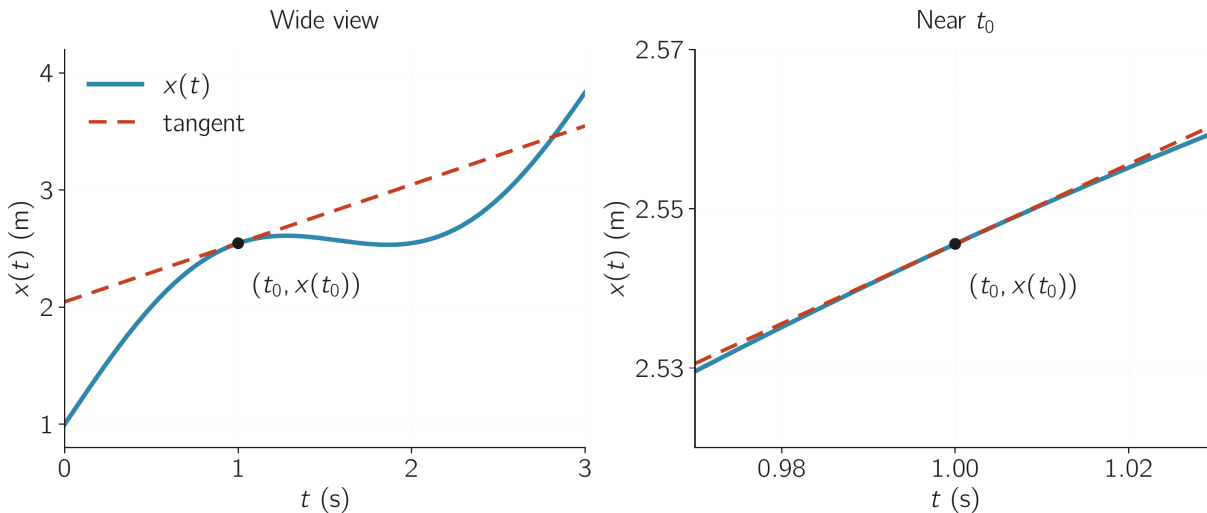


Figure 9: A smooth trajectory (blue) and its tangent at $t_0 = 1$ s (red dashed). The right panel magnifies the same curve over 0.97–1.03 s: the tangent closely approximates position over this short interval. The example is $x(t) = 1 \text{ m} + (1 \text{ m/s})t + (0.6 \text{ m}) \sin[(2 \text{ s}^{-1})t]$, so $x(t_0) \approx 2.546 \text{ m}$ and $v(t_0) \approx 0.501 \text{ m/s}$. The two panels use different axis scales.

5 Acceleration: the Second Derivative

Acceleration is the rate of change of velocity:

$$a(t) \equiv \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

In dot notation, $a(t) = \ddot{x}(t)$, read as x -double-dot.

Acceleration has dimensions $[L/T^2]$; its SI unit is m/s^2 . Just as $dx = v(t) dt$ relates position and time differentials, $dv = a(t) dt$ relates velocity and time differentials.

The constant shifts of space and time origins considered above also leave acceleration unchanged: differentiating the unchanged velocity once more gives $\ddot{x}' = \ddot{x}$.

For $x(t) = \frac{1}{2}at^2$, we found $v(t) = at$. Differentiating gives $dv/dt = a$, confirming that a is the constant acceleration.

Slope of a velocity graph

A straight velocity graph rises from 4 m/s at $t = 0$ to 12 m/s at $t = 4$ s. Is its acceleration increasing, decreasing, or constant? Is it positive or negative?

The slope is constant and positive:

$$a = \frac{12 \text{ m/s} - 4 \text{ m/s}}{4 \text{ s} - 0} = 2 \text{ m/s}^2.$$

Dividing $v(t)$ by t would not give this slope because the line does not pass through the origin.

Geometrically, acceleration is the rate of change of the slope of the x - t graph. Positive $\ddot{x}$ means $x(t)$ curves upward (concave up); negative $\ddot{x}$ means concave down.

For a sufficiently smooth trajectory, acceleration supplies the next term:

$$x(t_0 + \Delta t) \approx x(t_0) + v(t_0)\Delta t + \frac{1}{2}a(t_0)(\Delta t)^2.$$

This is the second-order Taylor approximation. The factor $\frac{1}{2}$ ensures that differentiating twice gives $a(t_0)$. If acceleration is constant, the quadratic expression is exact throughout the interval where that assumption holds.

Σ Taylor expansion

For a function equal to its Taylor series near $t = 0$,

$$f(t) = a_0 + a_1t + a_2t^2 + \cdots = \sum_{k=0}^{\infty} a_k t^k,$$

with coefficients $a_k = f^{(k)}(0)/k!$. Here $f^{(k)}$ is the k th derivative with respect to t , $f^{(0)} = f$, and $k! = 1 \cdot 2 \cdots k$, with $0! = 1$. Thus

$$f(t) = f(0) + f'(0)t + \frac{f''(0)}{2!}t^2 + \frac{f'''(0)}{3!}t^3 + \cdots.$$

Here the expansion point is the time origin. About a general time t_0 , replace powers of t by powers of $\Delta t = t - t_0$ and evaluate the derivatives at t_0 .

Truncating after the t^n term gives the **n th-order Taylor approximation**, a polynomial. First order keeps the constant and linear terms; second order also keeps the quadratic term. Retain enough terms for the accuracy needed, within the interval where the series represents the function. *Further reading:* Kleppner and Kolenkow, 2nd ed., Note 1.2, pp. 37–38.

How small is a short time interval? Compare it with a time scale τ of the motion, and write the dimensionless step as $u = \Delta t/\tau$. For $|u| \ll 1$, higher powers become smaller: if $|u| = 0.1$, then $u^2 = 0.01$ and $|u|^3 = 0.001$. In the plotted example, the sine varies with angular frequency $\omega = 2\text{ s}^{-1}$, so we can take $\tau = 1/\omega = 0.5\text{ s}$. The coefficients also matter: the omitted terms, including their coefficients, must be small enough for the accuracy we need.

① Changing acceleration

The derivative $j = da/dt$ is called **jerk** . It measures how quickly acceleration changes. An abrupt start in a lift involves a rapid change in acceleration; both the acceleration and its rate of change matter for comfort.

For a sense of scale, a change of velocity from 0 to 27 m/s in 6 s corresponds to an average acceleration of 4.5 m/s^2 , about $0.46g$, where $g \approx 9.8\text{ m/s}^2$. A comparable velocity change over a much shorter interval means a much larger average acceleration: 50 m/s in 5 ms gives 10^4 m/s^2 .

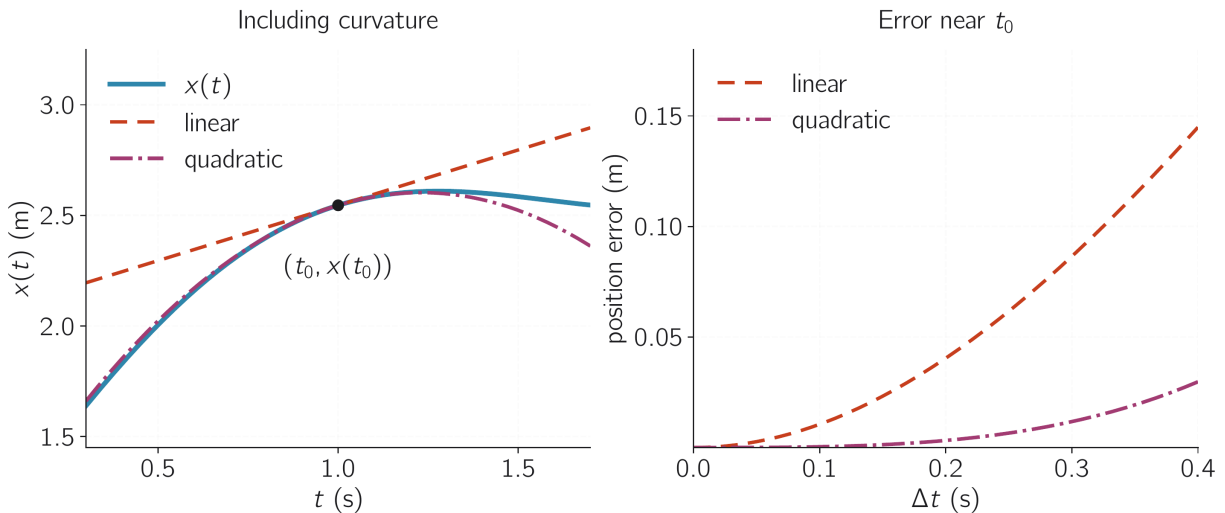


Figure 10: The same trajectory and expansion point $t_0 = 1$ s as in the linear-approximation figure. Left: exact position (blue), linear approximation (red dashed), and quadratic approximation (magenta dash-dot). The quadratic adds $\frac{1}{2}a(t_0)(t - t_0)^2$, with $a(t_0) \approx -2.182 \text{ m/s}^2$. Right: absolute position errors at $t = t_0 + \Delta t$ for positive time steps. Near t_0 , halving the time step reduces the linear error by about four and the quadratic error by about eight.

5.1 Constant acceleration

In one dimension, **motion at constant acceleration** satisfies one differential equation:

$$\ddot{x}(t) = a, \quad a = \text{const.}$$

The slope of $x(t)$ changes at a constant rate. Near Earth's surface, neglecting air resistance, vertical free fall has $a = -g$ if the upward direction is positive. We use y for the vertical coordinate.

∴ Guess the function, then check it

What function gives a constant after two derivatives? Try a quadratic with three unknown constant coefficients:

$$x_{\text{guess}}(t) = \alpha t^2 + \beta t + \gamma.$$

Differentiate twice:

$$\dot{x}_{\text{guess}} = 2\alpha t + \beta, \quad \ddot{x}_{\text{guess}} = 2\alpha.$$

To satisfy $\ddot{x} = a$, we need $2\alpha = a$, so $\alpha = a/2$. The initial conditions then fix the other coefficients:

$$\dot{x}(0) = v_0 \Rightarrow \beta = v_0, \quad x(0) = x_0 \Rightarrow \gamma = x_0.$$

This is the general form of a trajectory in one dimension over any interval of constant acceleration. It describes a family of possible motions. For a particular problem, we choose the position and time origins and the positive direction, then specify the signed acceleration a and the initial values x_0 and v_0 . These select the particular trajectory.

The same result follows by integrating twice. The first integral gives velocity:

$$v(t) = \int \ddot{x} dt = \int a dt = at + C_1.$$

The constant C_1 is the velocity at $t = 0$, so $C_1 = v_0$. Thus

$$v(t) = v_0 + at.$$

The second integral gives position:

$$x(t) = \int v(t) dt = \int (v_0 + at) dt = v_0 t + \frac{1}{2}at^2 + C_2.$$

The constant C_2 is the position at $t = 0$, so $C_2 = x_0$. Therefore

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2.$$

✓ Area under a velocity graph

The signed area under $v(t)$ gives displacement. For the positive v_0 and a shown, a rectangle contributes $v_0 t$ and a triangle contributes $\frac{1}{2}at^2$. Their sum is $x(t) - x_0$, not $x(t)$. The same algebra holds with signed contributions when v_0 or a is negative.

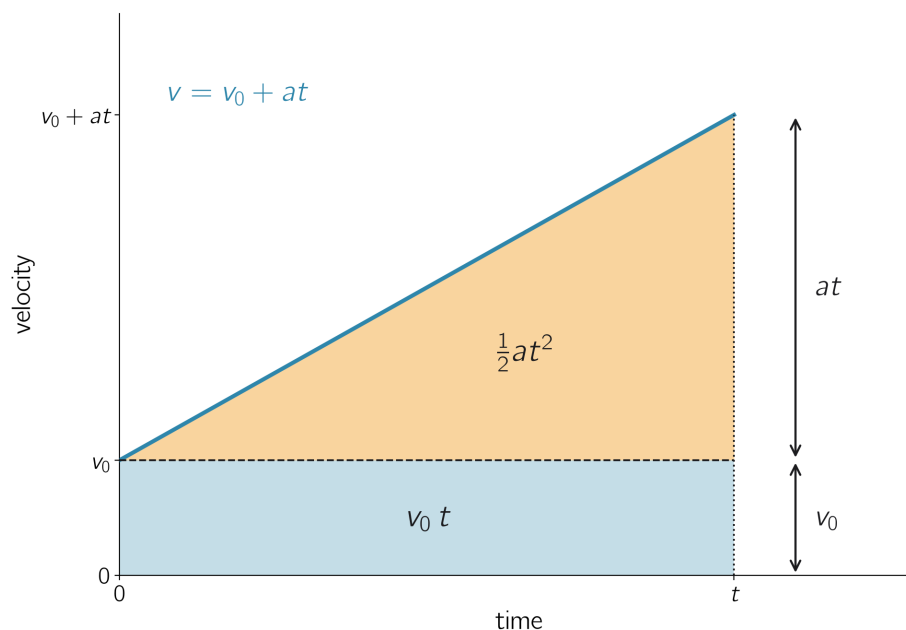


Figure 11: The area under the line $v = v_0 + at$ splits into a rectangle and a triangle. The rectangle contributes $v_0 t$ and the triangle contributes $\frac{1}{2}at^2$. Their sum is the displacement over the interval from 0 to t .

Useful corollary: v^2 in terms of x . Eliminating t between the two solutions gives

$$v^2 = v_0^2 + 2a(x - x_0).$$

This relation determines the speed $|v|$ at a position without finding the time. The sign of v must be determined from the motion, for example whether the particle is rising or falling.

5.2 The three motion classes

The three motion classes introduced in §1 can now be stated as differential equations:

Motion	Differential equation	Solution after initial data
At rest	$\dot{x} = 0$	$x(t) = x_0$
Constant velocity	$\dot{x} = v_0, v_0 = \text{const}$	$x(t) = x_0 + v_0 t$
Constant acceleration	$\ddot{x} = a, a = \text{const}$	$x(t) = x_0 + v_0 t + \frac{1}{2} a t^2$

For each row, the differential equation in the middle column and the initial conditions determine the solution in the last column.

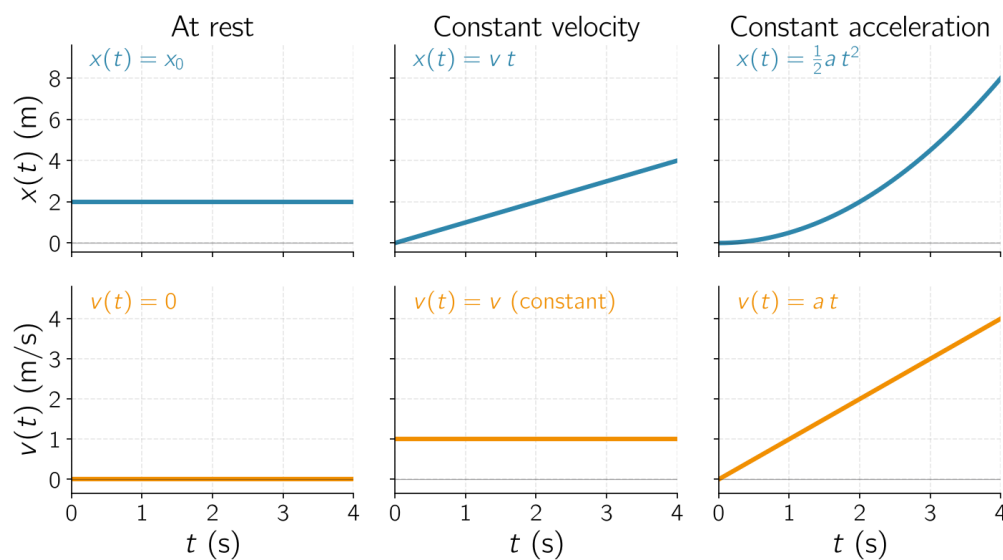


Figure 12: The position graphs from §1 now appear above their velocities. The bottom row is the derivative of the top row: zero for a constant position, constant for a linear position, and linear for a quadratic position.

6 Worked Example: Ball Thrown Upward

Choose the ground as $y = 0$, upward as positive, and $t = 0$ at release. A ball is thrown straight up from height $y_0 = 15$ m with initial velocity $v_0 = 10$ m/s. Neglect air resistance and take $g \approx 10$ m/s², so the acceleration is $a = -g = -10$ m/s².

✍ Before calculating

What is the sign of the acceleration while the ball rises, at the top, and while it falls?

Will the ball spend longer rising from its release point to the top, or falling from the top to the ground? Will its impact speed be greater than, equal to, or less than its launch speed?

Test the three signs in the **acceleration activity** ↗.

△ Path Through Space and Position Graph

The ball travels upward and downward along one vertical line. In panel (b) of the path-and-height figure, the horizontal axis is **time**, so the parabola shows height against time rather than the path through space. Check the axis labels before interpreting the shape.

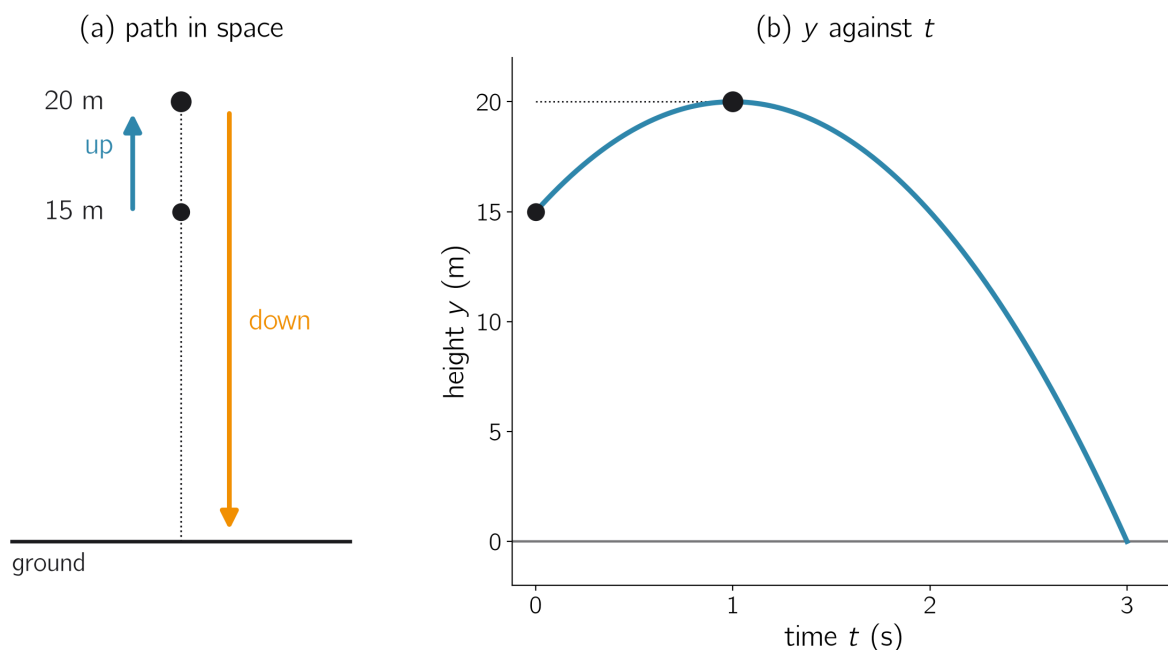


Figure 13: The two panels show the same throw. (a) In space the ball rises and falls along one vertical line from 15 m to 20 m and back to the ground. (b) On the graph the horizontal axis is time, so the same motion draws a parabola.

The equation of motion during free flight is $\ddot{y} = -g = -10 \text{ m/s}^2$. Solving it with $y(0) = 15 \text{ m}$ and $\dot{y}(0) = 10 \text{ m/s}$ gives, with t expressed in seconds,

$$v(t) = 10 - 10t \quad [\text{m/s}], \quad y(t) = 15 + 10t - 5t^2 \quad [\text{m}].$$

(a) Maximum height. Reached when $v(t^*) = 0$:

$$10 - 10t^* = 0 \quad \implies \quad t^* = 1 \text{ s}.$$

Plug into $y(t)$:

$$y(1) = 15 + 10(1) - 5(1)^2 = 20 \text{ m}. \quad \checkmark$$

△ Acceleration at the Apex

At the apex, the ball is momentarily at rest, but gravity acts unopposed throughout the flight, and the acceleration remains $a = -g = -10 \text{ m/s}^2$.

Velocity can be zero while acceleration is nonzero. Acceleration is the **slope** of $v(t)$, and a function can cross zero with a nonzero slope. Here $v(t) = 10 - 10t$ crosses zero at $t = 1 \text{ s}$ while its slope remains -10 m/s^2 .

The ball is not in equilibrium at the apex. Since $a = -g$, its velocity becomes negative just after the apex and the ball moves downward.

The acceleration is negative while rising, at the top, and while falling. Zero velocity at the top does not mean zero acceleration.

(b) When does it hit the ground? Set $y(t) = 0$:

$$15 + 10t - 5t^2 = 0 \quad \implies \quad t^2 - 2t - 3 = 0 \quad \implies \quad (t - 3)(t + 1) = 0.$$

✓ When is the equation of motion valid?

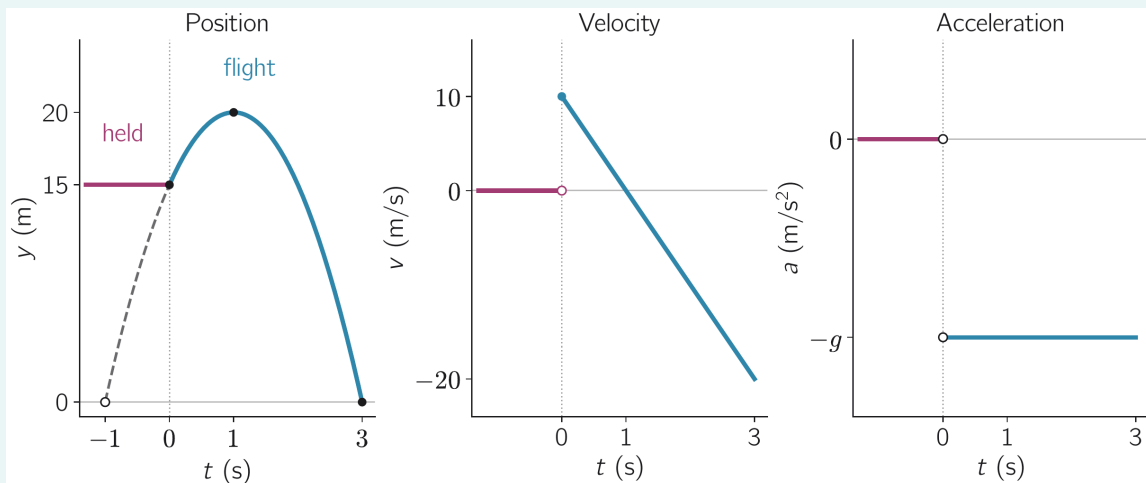
Always identify the interval over which the equation of motion describes the physical system.

Both $t = 3\text{ s}$ and $t = -1\text{ s}$ solve the quadratic. Which describes the actual throw? To make the stages explicit, idealize the launch as an instantaneous push at $t = 0$:

$$\ddot{y} = \begin{cases} 0, & t < 0 \quad (\text{held still}), \\ -g, & 0 < t < t_{\text{ground}} \quad (\text{free flight}). \end{cases}$$

Before launch, the hand supports the ball and its velocity is zero, so $y(t) = y_0$. The push changes its velocity from zero to v_0 ; position stays continuous. Thus

$$y(t) = \begin{cases} y_0, & t < 0, \\ y_0 + v_0 t - \frac{1}{2}gt^2, & 0 \leq t \leq t_{\text{ground}}. \end{cases}$$



The corner in position and the jump in velocity represent the idealized push. The acceleration panel shows the held and free-flight stages; it omits the instantaneous push itself. A real throw takes a short time, during which the hand accelerates the ball.

The dashed curve extends the free-flight parabola backward to its root at $t = -1\text{ s}$. This is not a ground crossing in the actual motion: the ball was held at y_0 , and the free-flight rule was not valid then. The root $t = 3\text{ s}$ is the first impact after release. At impact the ground changes the motion, so the free-flight rule also stops applying there.

Changing the clock's zero changes the signs of times, not the interval in which the rule is valid.

Distance, then speed. Displacement has a sign. On a stretch without a reversal, the distance travelled is

$$\Delta s = \sqrt{(\Delta y)^2} = |\Delta y| = |y_f - y_i|.$$

If the motion reverses, add the distances of the separate stretches. Here the ball travels 5 m up and 20 m down: the total distance is 25 m, while the displacement is -15 m .

Let $s(t)$ be the distance accumulated since release, with $s(0) = 0$. Speed is its rate of change:

$$\text{instantaneous speed} = \dot{s}(t) = \frac{ds}{dt} = \left| \frac{dy}{dt} \right| = |v(t)|.$$

For forward time steps, $ds = |dy| = |v(t)| dt$: take the length of each small displacement before adding. Over a finite interval,

$$\Delta s = \int_{t_i}^{t_f} |v(t)| dt, \quad \Delta y = \int_{t_i}^{t_f} v(t) dt.$$

The distances add even when the velocity changes sign; signed displacements can cancel. Thus instantaneous speed is $|v(t)|$, but average speed $\Delta s / \Delta t$ need not equal $|\Delta y / \Delta t|$.

Distance never decreases; velocity tells us the direction of motion. With upward positive, $v > 0$ means rising and $v < 0$ means falling. Reversing the coordinate axis changes the signs of displacement, velocity, and acceleration, but leaves distance and speed unchanged. Next class we extend this description to motion in two dimensions, where the direction of motion can also change.

(c) Speed on impact. $v(3) = 10 - 10(3) = -20$ m/s. The minus sign means downward; the ball is falling. The speed is 20 m/s.

△ Acceleration and Speed

The acceleration is -10 m/s^2 for all three seconds. The ball slows during the first second and speeds up during the next two. The sign of a gives the direction in which velocity changes. Speed decreases when v and a have opposite signs and increases when they have the *same* sign. The same constant acceleration -10 m/s^2 therefore produces both parts of the motion.

✓ Checks

- Dimensions: $v_0 t$ and gt^2 are lengths; gt is a velocity.
- Final speed: from $v^2 = v_0^2 + 2a(y - y_0) = 100 + 2(-10)(0 - 15) = 400$, so $|v| = 20$ m/s. ✓
- The descent lasts 2 s, longer than the 1 s ascent. Impact speed is greater than launch speed because the ground is below the release point.

⊙ Check out the animation

Open the animation ↗ Two beaded ropes are released from rest. Beads at heights $h, 4h, 9h, 16h$ above the floor hit at equal time intervals: the fall time from height H is $t = \sqrt{2H/g}$.

7 From Kinematics to Dynamics

So far, we have described motion by assuming constant velocity or constant acceleration. In **dynamics**, we ask which physical laws govern the motion and use them to write an equation for how the velocity changes at each instant.

Newton's law relates acceleration to the net force:

$$\ddot{x} = \frac{F_{\text{net}}}{m}.$$

With the initial position and velocity, we can use calculus to solve this differential equation and find the entire trajectory $x(t)$, following the same steps we used for the falling ball.

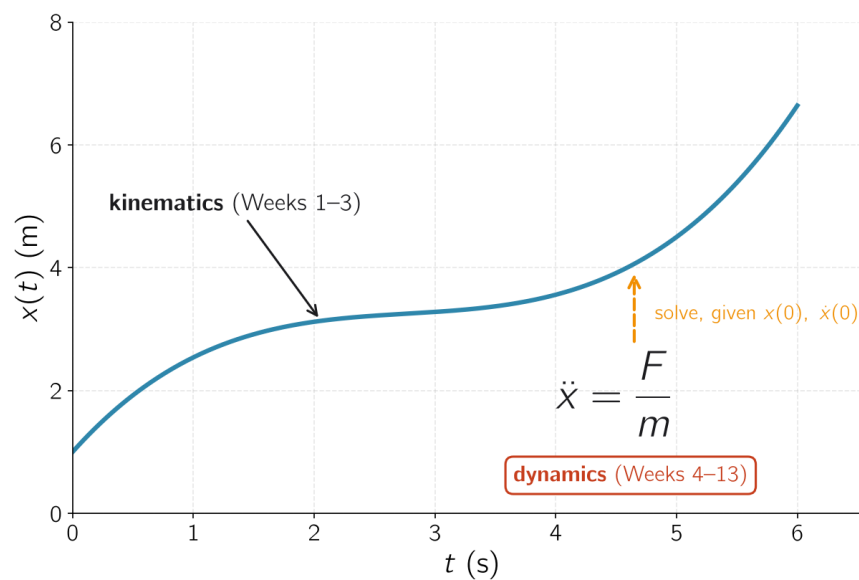


Figure 14: Kinematics describes position $x(t)$, velocity $\dot{x}(t)$, and acceleration $\ddot{x}(t)$. Dynamics relates acceleration to the net force F and mass m through $\ddot{x} = F/m$. Given a force law and initial data, solving that equation gives the trajectory.

Our goal as physicists is to understand the simple rules of nature and identify the equations of motion that apply at each instant.

Summary

Quantity	Symbol	Definition	Units
Position	$x(t)$	Signed coordinate relative to a chosen origin and positive direction	m
Displacement	Δx	$x(t_f) - x(t_i)$	m
Distance travelled	$s(t)$	Accumulated path length since the initial time	m
Average velocity	$\bar{v}$	$\Delta x / \Delta t$	m/s
Instantaneous velocity	$v(t) = \dot{x}$	dx/dt	m/s
Instantaneous speed	$\dot{s}$	$ds/dt = v(t) $	m/s
Acceleration	$a(t) = \ddot{x}$	$dv/dt = d^2x/dt^2$	m/s ²