

Physics I: Mechanics and Relativity

Motion in 2D/3D and Vector Algebra

Week 02

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Readings

1. **Feynman Lectures I, Ch. 11, §§11-4–11-7.** Vector definitions, addition, components and the scalar product. [feynmanlectures.caltech.edu/I_11.html](https://www.feynmanlectures.caltech.edu/I_11.html) ↗
2. **Kleppner & Kolenkow, §§1.2–1.9.** Vector algebra, components, the position vector, and motion in several dimensions.

Overview

Last week we described position, velocity and acceleration along a line. A thrown ball moves horizontally while rising and falling, so one coordinate is no longer enough to describe its motion. Vectors let us describe these motions together. With constant downward gravity and no air resistance, the horizontal and vertical equations can be solved separately using the same time.

1 From One to Several Dimensions

In one dimension, position $x(t)$ is a function of time. Velocity $v(t) = \dot{x}$ is its derivative, and acceleration $a(t) = \ddot{x}$ is its second derivative. The dot is Newton's shorthand for a time derivative: $\dot{x} \equiv dx/dt$ and $\ddot{x} \equiv d^2x/dt^2$. Constant acceleration was defined by the differential equation

$$\ddot{x} = a, \quad a = \text{const.}$$

Solving it with the initial position and velocity gives

$$v(t) = v_0 + at, \quad x(t) = x_0 + v_0t + \frac{1}{2}at^2.$$

Here $x_0 = x(0)$ and $v_0 = v(0)$. A single coordinate describes motion along a line; a ball thrown at an angle needs two coordinates.

In several dimensions, position, velocity and acceleration are vectors. Differentiation acts on each component. Vector addition combines displacements, and projections separate motion along perpendicular directions.

2 What is a Vector?

A **vector** carries two pieces of information: a **magnitude** (how big) and a **direction** (which way), except that the zero vector has no direction. Velocity is a vector: 30 m/s northeast is not the same as 30 m/s straight down. Force and displacement are vectors too. Temperature, mass, and time are **scalars**: they are specified by a single value and units, without a spatial direction.

△ Scalars and constants

A scalar has no direction, but its value may change. The temperature $T(t)$ in a room can rise during the day. **Scalar** describes the kind of quantity; **constant** describes whether its value changes.

Boldface $\mathbf{A}$ and an arrow $\vec{A}$ are two notations for the same vector. A unit vector is both bold and hatted, as in $\hat{\mathbf{x}}$. A magnitude is always non-negative, and we write it $|\mathbf{A}|$ or just A .

For vector algebra, the arrow has no fixed starting point. We can slide it anywhere on the page: as long as its length and direction stay the same, it represents the same vector.

Pick two perpendicular unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ along your chosen axes. Unit means $|\hat{\mathbf{x}}| = |\hat{\mathbf{y}}| = 1$; orthogonal means $\hat{\mathbf{x}} \perp \hat{\mathbf{y}}$. Any vector in the plane can then be written

$$\mathbf{A} = A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}},$$

where the **components** A_x, A_y are quantities with the same units as $\mathbf{A}$. In 3D we add a third orthogonal axis $\hat{\mathbf{z}}$ and write $\mathbf{A} = A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}}$.

Other books write the same basis vectors as $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$, and we will often write $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ when an index is convenient. The names are the writer's choice: once $\hat{\mathbf{i}}$ is declared to point along x , $\hat{\mathbf{i}} = \hat{\mathbf{x}}$. State which direction each symbol points, then keep to one set of names within a calculation.

The magnitude is the length of the arrow, obtained from the Pythagorean theorem:

$$|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}.$$

3 Vector Algebra

3.1 Addition and scalar multiplication

Vectors add **component by component**:

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}}.$$

Geometrically, vectors add tip-to-tail. Place the tail of $\mathbf{B}$ at the head of $\mathbf{A}$, and the sum is the arrow from the original tail to the final head. Vector addition is commutative ($\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$) and associative: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

Scalar multiplication $\lambda \mathbf{A}$ stretches the arrow by $|\lambda|$ and reverses it if $\lambda < 0$:

$$\lambda \mathbf{A} = (\lambda A_x)\hat{\mathbf{x}} + (\lambda A_y)\hat{\mathbf{y}} + (\lambda A_z)\hat{\mathbf{z}}.$$

Scalar multiplication is **distributive**: for scalars λ and μ ,

$$\lambda(\mathbf{A} + \mathbf{B}) = \lambda\mathbf{A} + \lambda\mathbf{B}, \quad (\lambda + \mu)\mathbf{A} = \lambda\mathbf{A} + \mu\mathbf{A}.$$

Multiplying a sum of vectors scales each vector; adding two scalar coefficients gives the same result as scaling the vector separately and adding. These rules follow component by component from ordinary scalar algebra: for example, $\lambda(A_x + B_x) = \lambda A_x + \lambda B_x$.

Subtraction is addition of the negative: $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$. The vector $\mathbf{A} - \mathbf{B}$ points *from* the head of $\mathbf{B}$ *to* the head of $\mathbf{A}$, when their tails are drawn at the same point.

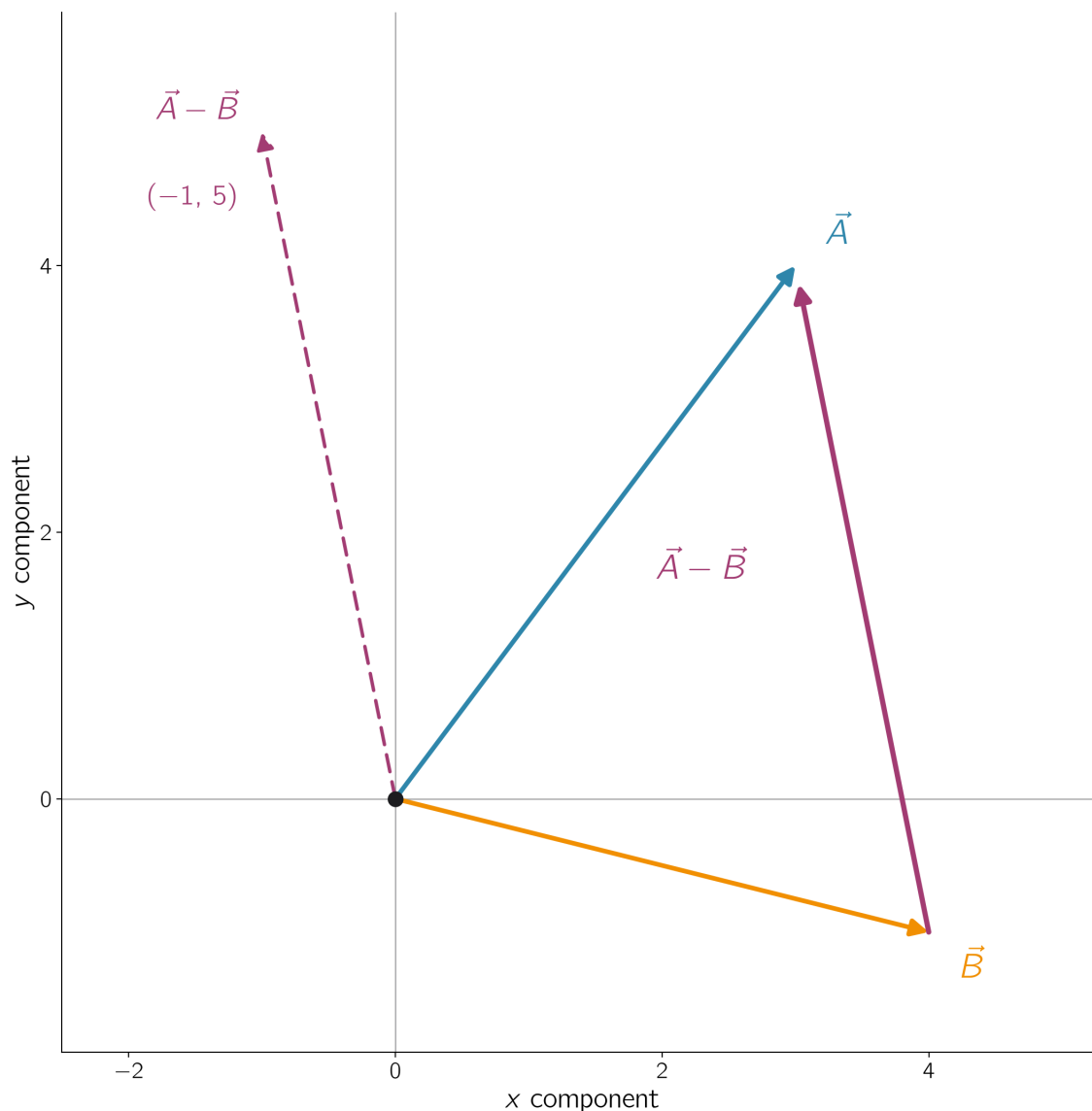


Figure 1: Both $\mathbf{A} = 3\hat{\mathbf{x}} + 4\hat{\mathbf{y}}$ and $\mathbf{B} = 4\hat{\mathbf{x}} - \hat{\mathbf{y}}$ start at the origin. The solid magenta arrow runs from the head of $\mathbf{B}$ to the head of $\mathbf{A}$, and that arrow is $\mathbf{A} - \mathbf{B}$. The dashed copy carries the same vector back to the origin, where its components read $(-1, 5)$.

✍ Adding two vectors

Take $\mathbf{A} = 3\hat{x} + 4\hat{y}$ and $\mathbf{B} = 4\hat{x} - \hat{y}$. Sketch their sum and estimate its magnitude. How does it compare with $|\mathbf{A}| + |\mathbf{B}|$?

The individual magnitudes are $|\mathbf{A}| = 5$ and $|\mathbf{B}| = \sqrt{17} \approx 4.1$. Adding components gives $\mathbf{A} + \mathbf{B} = 7\hat{x} + 3\hat{y}$, with magnitude $\sqrt{58} \approx 7.6$, less than $5 + \sqrt{17} \approx 9.1$. The angle between the vectors affects the magnitude of their sum. In general,

$$|\mathbf{A} + \mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}|.$$

Equality holds when nonzero vectors point in the same direction, or when either vector is zero.

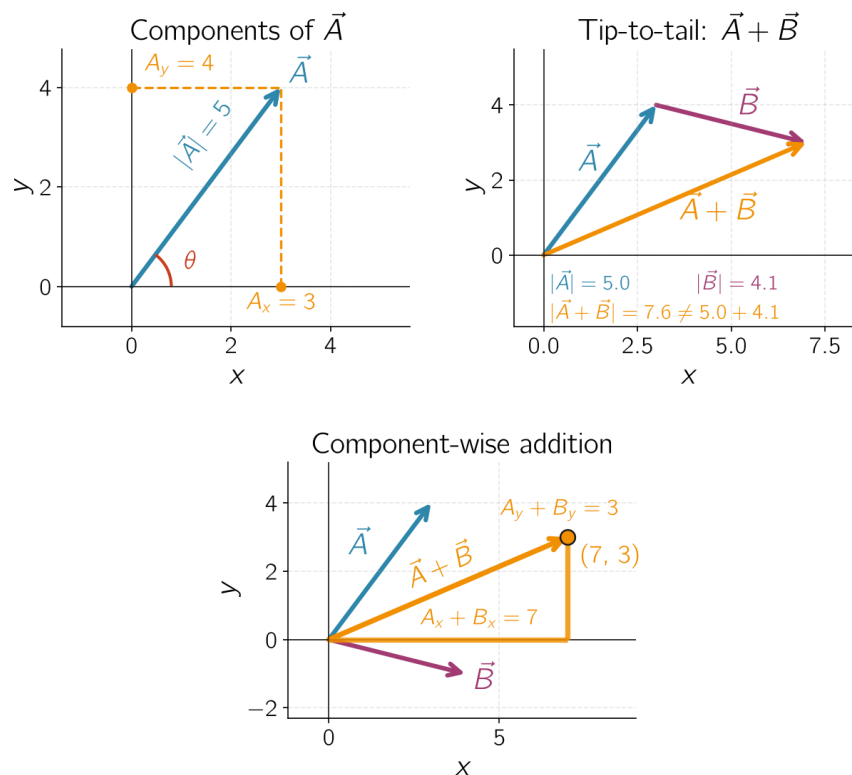


Figure 2: Anatomy of a vector. Top left: $\mathbf{A}$ with magnitude $|\mathbf{A}|$ and angle θ from the x -axis, so $A_x = |\mathbf{A}| \cos \theta$ and $A_y = |\mathbf{A}| \sin \theta$. Top right: tip-to-tail addition, with $|\mathbf{A}| = 5.0$, $|\mathbf{B}| = 4.1$ and $|\mathbf{A} + \mathbf{B}| = 7.6$. Below: the same sum from components, $\mathbf{A} + \mathbf{B} = 7\hat{x} + 3\hat{y}$.

✓ Adding displacements

Walk east, then north. The net displacement is the arrow from your starting point to your endpoint. Each leg contributes to one coordinate, which gives the same result as tip-to-tail addition. This geometric rule does not by itself imply that the equations of motion along different axes are independent.

The unit vector along a given vector. Any nonzero vector $\mathbf{A}$ defines a unit vector in its direction: divide by the magnitude,

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|}, \quad |\hat{\mathbf{A}}| = 1.$$

This is scalar multiplication by $1/|\mathbf{A}|$, so the direction is unchanged and the length becomes one. For example, $\hat{\mathbf{x}} + \hat{\mathbf{y}}$ points along the diagonal but has magnitude $\sqrt{1^2 + 1^2} = \sqrt{2}$, so it is not a unit vector; the unit vector along the diagonal is $(\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$. The number $\sqrt{2}$ is the magnitude, not the unit vector: a unit vector is a vector and has a direction. Conversely, every nonzero vector is its magnitude times its direction, $\mathbf{A} = |\mathbf{A}| \hat{\mathbf{A}}$.

3.2 Velocity and falling balls

The three coordinates are collected into one vector. The **position vector** of a particle at time t is

$$\mathbf{r}(t) = x(t) \hat{\mathbf{x}} + y(t) \hat{\mathbf{y}} + z(t) \hat{\mathbf{z}}.$$

By convention this is drawn from the origin to the particle.

Velocity is the time derivative of position, taken **component by component**. The unit vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are *constants* in this fixed Cartesian basis, so they pass straight through the derivative:

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \dot{x} \hat{\mathbf{x}} + \dot{y} \hat{\mathbf{y}} + \dot{z} \hat{\mathbf{z}}.$$

Each component of velocity is the time derivative of the corresponding position component. The same recipe gives **acceleration**:

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \ddot{x} \hat{\mathbf{x}} + \ddot{y} \hat{\mathbf{y}} + \ddot{z} \hat{\mathbf{z}}.$$

Week 1 wrote the displacement in a short time dt as $dx = v dt$. In the plane the particle moves $dx = \dot{x} dt$ along x and $dy = \dot{y} dt$ along y in the same interval, and the two need not be equal. The displacement $d\mathbf{r} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}}$ is a vector; dividing by the scalar dt gives $\mathbf{v}$. In $\mathbf{v} = \dot{x} \hat{\mathbf{x}} + \dot{y} \hat{\mathbf{y}}$ the components $\dot{x}$ and $\dot{y}$ are scalars and the basis vectors carry the direction. The vector equation packages the two scalar equations $v_x = \dot{x}$ and $v_y = \dot{y}$. Together the components determine both the speed, $|\mathbf{v}| = \sqrt{\dot{x}^2 + \dot{y}^2}$, and, when the speed is nonzero, the direction $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$.

✍ Dropped and horizontally launched balls

Two identical balls are released from the same height at the same instant. One is dropped; the other is fired **horizontally** at 20 m/s. Neglect air resistance. Which reaches level ground first, and why?

The **drop-and-shoot demonstration** ↗ compares the two motions from release.

Over heights small compared with Earth's radius, model gravity as a constant downward acceleration. Neglect air resistance and Earth's rotation. With $\hat{\mathbf{y}}$ upward, the equation is

$$\boxed{\ddot{\mathbf{r}} = -g \hat{\mathbf{y}}} \quad \Longleftrightarrow \quad \ddot{x} = 0, \quad \ddot{y} = -g.$$

Here $g \approx 9.8 \text{ m/s}^2$. Equality of vectors requires equality of their components: the right-hand side has zero component along $\hat{x}$ and component $-g$ along $\hat{y}$. Thus each coordinate has its own equation to solve. The horizontal equation gives constant velocity, while the vertical equation is the free fall of Week 1. The vertical motion has the same form whatever the initial horizontal velocity.

Put the origin on the ground below the release point, and write h for the release height. At $t = 0$, both balls have $y = h$ and $v_y = 0$. Let u be the initial horizontal velocity: $u = 0$ for the dropped ball and $u = 20 \text{ m/s}$ for the launched ball. Solving the two component equations gives

$$x(t) = ut, \quad y(t) = h - \frac{1}{2}gt^2,$$

$$\mathbf{v}(t) = u\hat{x} - gt\hat{y}.$$

The velocity is a sum of two vectors, each multiplied by a scalar. Its horizontal component stays constant while its vertical component changes. Setting $y = 0$ gives the landing time $t_f = \sqrt{2h/g}$, which does not depend on u . For $h = 20 \text{ m}$, $t_f \approx 2.02 \text{ s}$.

Gravity gives the two balls identical vertical motion: they start at the same height with the same vertical velocity and have the same downward acceleration. They therefore land together. The horizontal launch changes where the fired ball lands, not when it lands.

⚠ Horizontal motion and falling

A ball rolling off a table starts falling immediately after it loses contact with the table. It does not first travel horizontally until its forward motion is spent.

3.3 Throwing the ball in two dimensions

Return to the ball thrown upward in Week 1, but now give it a horizontal velocity too. Release it at $t = 0$ from $(0, h)$, with initial velocity

$$\mathbf{v}_0 = v_{0x}\hat{x} + v_{0y}\hat{y}.$$

Here v_{0x} and v_{0y} are the initial velocity components. With upward positive, gravity still gives

$$\ddot{x} = 0, \quad \ddot{y} = -g.$$

What functions have these second derivatives? The horizontal position is linear in time and the vertical position is quadratic. Fixing their constants from the initial position and velocity gives

$$\begin{aligned} x(t) &= v_{0x}t, & v_x(t) &= v_{0x}, \\ y(t) &= h + v_{0y}t - \frac{1}{2}gt^2, & v_y(t) &= v_{0y} - gt. \end{aligned}$$

The vertical equation is exactly the one from last class. The same time t goes into both coordinates: together, $x(t)$ and $y(t)$ locate the ball at that instant. In vector form,

$$\mathbf{r}(t) = v_{0x}t\hat{x} + \left(h + v_{0y}t - \frac{1}{2}gt^2\right)\hat{y}, \quad \mathbf{v}(t) = v_{0x}\hat{x} + (v_{0y} - gt)\hat{y}.$$

Take $v_{0x} > 0$ and $v_{0y} > 0$, so the ball initially moves to the right and upward. Its vertical velocity decreases through zero at

$$t_* = \frac{v_{0y}}{g}.$$

Before t_* the height increases; afterward it decreases. This is the highest point. At that instant the horizontal motion continues:

$$v_y(t_*) = 0, \quad v_x(t_*) = v_{0x}, \quad \mathbf{v}(t_*) = v_{0x} \hat{\mathbf{x}}.$$

At the highest point the vertical component of velocity is zero, not the velocity vector. The speed has its minimum value $|\mathbf{v}(t_*)| = v_{0x}$. For the purely vertical throw of Week 1, $v_{0x} = 0$ as well, and only then does the ball momentarily stop. In both cases the acceleration remains $-g\hat{\mathbf{y}}$ at the top; zero vertical velocity does not mean zero vertical acceleration.

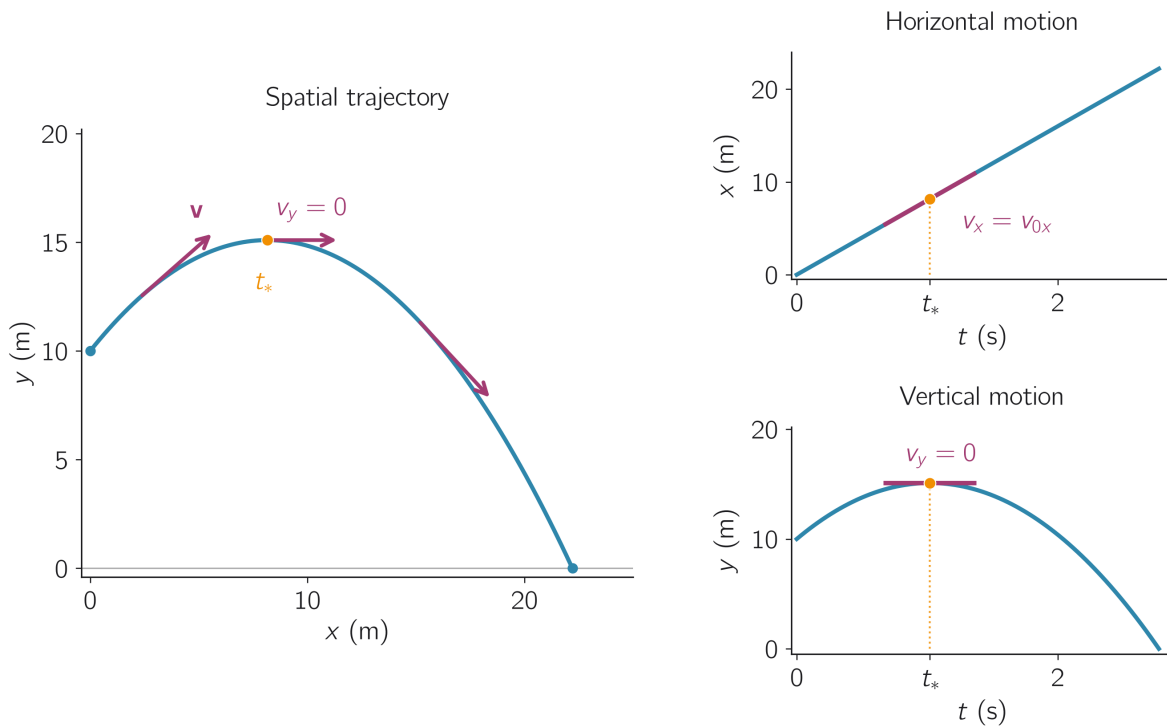


Figure 3: One throw, shown as a path $y(x)$ and as the two coordinate functions $x(t)$ and $y(t)$. Here $h = 10$ m, $v_{0x} = 8$ m/s, $v_{0y} = 10$ m/s and $g = 9.8$ m/s². The marked instant is $t_* = v_{0y}/g \approx 1.02$ s. Arrows on the spatial path represent velocity, drawn with a common scale; their lengths are not distances traveled.

Why is velocity tangent to the spatial path? In a graph of $y(t)$, the derivative gives the slope v_y . Figure 3 also shows a different curve: the path in the xy plane, with no time axis. Why should the velocity vector point along the tangent to this curve?

Choose a point $\mathbf{r}(t)$ on the path and a second point reached a short time later. The displacement arrow joining them lies along their secant line. Dividing this displacement by the elapsed time

rescales the arrow without changing its direction. As the time interval shrinks to zero, the secant approaches the tangent and the average velocity approaches the instantaneous velocity. In the differential notation of Week 1,

$$d\mathbf{r} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}}, \quad dx = v_x(t) dt, \quad dy = v_y(t) dt.$$

Combining the two components gives

$$d\mathbf{r} = (v_x(t)\hat{\mathbf{x}} + v_y(t)\hat{\mathbf{y}})dt = \mathbf{v}(t) dt.$$

For $dt > 0$, the differential displacement $d\mathbf{r}$ and the velocity $\mathbf{v}$ point in the same direction. The differentials describe the local tangent: for a small finite time step, they approximate the actual displacement,

$$\mathbf{r}(t + dt) \simeq \mathbf{r}(t) + d\mathbf{r} = \mathbf{r}(t) + \mathbf{v}(t) dt.$$

With t fixed, the right-hand side describes a straight line through $\mathbf{r}(t)$ in the direction $\mathbf{v}(t)$: the tangent line. This argument applies to any smooth motion, wherever $\mathbf{v}(t) \neq 0$. The velocity arrows in Figure 3 are placed at the corresponding positions to show the direction and speed there; they are not position vectors. If the velocity is zero, it has no direction, so this argument does not select a tangent direction at that instant.

△ The path and the position–time graphs

On the spatial plot $y(x)$, both axes are positions. Its slope is not a velocity: for $v_x \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{v_y}{v_x}.$$

At the apex this ratio is zero, so the tangent is horizontal. The velocity arrow is horizontal too, with nonzero length.

On the plot $y(t)$, the horizontal axis is time and the slope is v_y . Its horizontal tangent at t_* means that the height is momentarily not changing to first order. On the plot $x(t)$, the slope is $v_x = v_{0x} > 0$, including at t_* : this graph is a straight rising line and has no maximum.

The two horizontal tangents, in $y(x)$ and $y(t)$, describe the same event, but their slopes mean different things. Neither says that the ball has stopped. To have zero velocity, both $\dot{x}$ and $\dot{y}$ must vanish at the same time.

4 The dot product (scalar product)

So far, we have added vectors and multiplied them by scalars. What does it mean to multiply two vectors? We need to specify which product we mean: the result can be a scalar or a vector.

The **dot product**, also called the **inner product** or **scalar product**, gives a scalar. It measures how much one vector points along the other, weighted by their magnitudes. The **cross product**, or **vector product**, gives a vector; we introduce it in the next section.

Choosing axes along one vector. Keep the vectors fixed and choose the positive x axis along a nonzero $\mathbf{A}$. Choose the xy plane to contain both vectors, with $\mathbf{B}$ on the nonnegative y side. If θ is the angle between them, then

$$\mathbf{A} = A\hat{\mathbf{x}}, \quad \mathbf{B} = B \cos \theta \hat{\mathbf{x}} + B \sin \theta \hat{\mathbf{y}}, \quad A = |\mathbf{A}|, \quad B = |\mathbf{B}|.$$

In these axes, multiplying matching components and adding gives

$$A_x B_x + A_y B_y = A(B \cos \theta) + 0(B \sin \theta) = AB \cos \theta.$$

This is the geometric dot-product formula, $\mathbf{A} \cdot \mathbf{B} = AB \cos \theta$. The choice of axes makes its meaning visible: only the component of $\mathbf{B}$ along $\mathbf{A}$ contributes. Below we verify that the same component sum is obtained when the axes are rotated.

Projection. Draw a perpendicular from the tip of $\mathbf{B}$ to the line along $\mathbf{A}$. The part along that line is the vector projection,

$$\mathbf{B}_{\parallel} = (B \cos \theta) \hat{\mathbf{A}}, \quad \hat{\mathbf{A}} = \frac{\mathbf{A}}{A}, \quad \mathbf{B}_{\perp} = \mathbf{B} - \mathbf{B}_{\parallel}.$$

The component $B \cos \theta$ is positive along $\hat{\mathbf{A}}$ and negative in the opposite direction. It is a scalar; multiplying by $\hat{\mathbf{A}}$ makes the vector projection. The dot product includes the extra factor A :

$$\mathbf{B} \cdot \hat{\mathbf{A}} = B \cos \theta, \quad \mathbf{A} \cdot \mathbf{B} = A(\mathbf{B} \cdot \hat{\mathbf{A}}).$$

We can instead project $\mathbf{A}$ along $\mathbf{B}$. That component is $A \cos \theta$, and multiplying it by B gives the same result:

$$A(B \cos \theta) = B(A \cos \theta), \quad \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}.$$

The projected components are generally different; the dot products are equal. For an acute angle the dot product is positive, for a right angle it is zero, and for an obtuse angle it is negative. Check: for $\mathbf{B} = 2\mathbf{A}$ the angle is zero and $\mathbf{A} \cdot \mathbf{B} = 2A^2$; for $\mathbf{B} = -2\mathbf{A}$ the angle is π and $\mathbf{A} \cdot \mathbf{B} = -2A^2$. A zero vector has zero dot product with every vector.

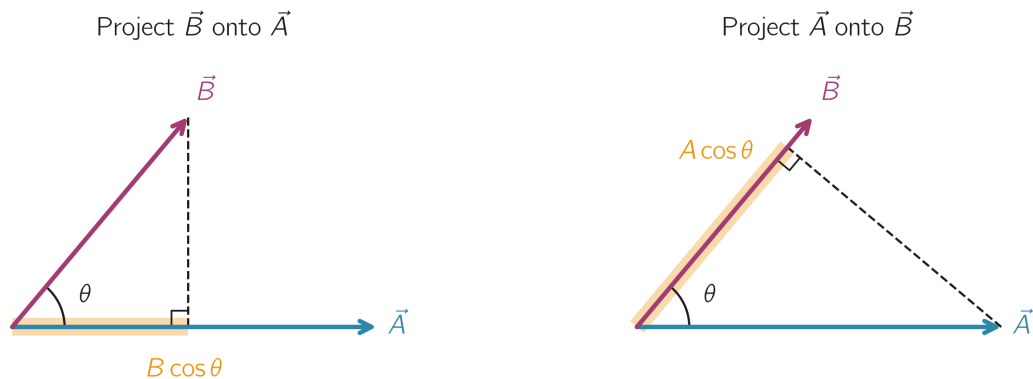


Figure 4: The same vectors are shown in both panels: $\mathbf{A}$ in blue and $\mathbf{B}$ in magenta. The orange segments are $B \cos \theta$ along $\mathbf{A}$ (left) and $A \cos \theta$ along $\mathbf{B}$ (right). Multiplying the left projection by A , or the right projection by B , gives the same scalar $AB \cos \theta$.

Magnitude and direction. What happens when we take the dot product of a vector with itself? The angle is zero, so

$$\mathbf{A} \cdot \mathbf{A} = A^2 \cos 0 = A^2.$$

The magnitude is therefore the square root of this inner product, and the unit vector in its direction is

$$A = \sqrt{\mathbf{A} \cdot \mathbf{A}}, \quad \hat{\mathbf{A}} = \frac{\mathbf{A}}{A} = \frac{\mathbf{A}}{\sqrt{\mathbf{A} \cdot \mathbf{A}}} \quad (A \neq 0).$$

Since $\hat{\mathbf{A}}$ has magnitude one, $\hat{\mathbf{A}} \cdot \hat{\mathbf{A}} = 1$.

Perpendicular vectors. If two nonzero vectors are perpendicular, $\theta = \pi/2$ and $\cos \theta = 0$, so their dot product vanishes. Conversely, a zero dot product between nonzero vectors means they are perpendicular. For the Cartesian unit vectors this gives

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1, \quad \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0.$$

Choosing a basis. A basis is a set of vectors from which every vector in the space can be written uniquely as a linear combination. In three dimensions we need three linearly independent vectors: none can be written as a combination of the other two. In two dimensions, two nonparallel vectors are enough.

Our Cartesian unit vectors form an **orthonormal basis**: they are mutually perpendicular (orthogonal), and each has magnitude one (normalized). A basis does not have to be orthonormal, but this choice makes the components especially easy to find. Dotting $\mathbf{A} = A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}}$ with each basis vector gives

$$A_x = \mathbf{A} \cdot \hat{\mathbf{x}}, \quad A_y = \mathbf{A} \cdot \hat{\mathbf{y}}, \quad A_z = \mathbf{A} \cdot \hat{\mathbf{z}}.$$

Each component is just the projection along its basis direction.

For example,

$$\begin{aligned} \mathbf{A} \cdot \hat{\mathbf{x}} &= (A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}}) \cdot \hat{\mathbf{x}} \\ &= A_x \underbrace{\hat{\mathbf{x}} \cdot \hat{\mathbf{x}}}_1 + A_y \underbrace{\hat{\mathbf{y}} \cdot \hat{\mathbf{x}}}_0 + A_z \underbrace{\hat{\mathbf{z}} \cdot \hat{\mathbf{x}}}_0 = A_x. \end{aligned}$$

The same calculation gives the y and z components.

From geometry to components. We can now obtain the component formula. The dot product distributes over addition, and scalar factors can be taken outside it:

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}, \quad (\lambda \mathbf{A}) \cdot \mathbf{B} = \lambda(\mathbf{A} \cdot \mathbf{B}).$$

Write $\mathbf{A} = A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}}$ and $\mathbf{B} = B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}$. Expanding gives

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= (A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}}) \cdot (B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}) \\ &= A_x B_x \underbrace{\hat{\mathbf{x}} \cdot \hat{\mathbf{x}}}_1 + A_x B_y \underbrace{\hat{\mathbf{x}} \cdot \hat{\mathbf{y}}}_0 + A_x B_z \underbrace{\hat{\mathbf{x}} \cdot \hat{\mathbf{z}}}_0 \\ &\quad + A_y B_x \underbrace{\hat{\mathbf{y}} \cdot \hat{\mathbf{x}}}_0 + A_y B_y \underbrace{\hat{\mathbf{y}} \cdot \hat{\mathbf{y}}}_1 + A_y B_z \underbrace{\hat{\mathbf{y}} \cdot \hat{\mathbf{z}}}_0 \\ &\quad + A_z B_x \underbrace{\hat{\mathbf{z}} \cdot \hat{\mathbf{x}}}_0 + A_z B_y \underbrace{\hat{\mathbf{z}} \cdot \hat{\mathbf{y}}}_0 + A_z B_z \underbrace{\hat{\mathbf{z}} \cdot \hat{\mathbf{z}}}_1 \\ &= A_x B_x + A_y B_y + A_z B_z. \end{aligned}$$

The dot product is the sum of the products of matching components.

Every mixed term vanishes because the basis directions are perpendicular; each matching term has a unit-vector dot product equal to one. In two dimensions, simply omit the z terms. Rotating the axes changes the components, but it changes neither the magnitudes nor the angle between the vectors, so the dot product is unchanged.

↗ Beyond the basics: Einstein notation and the Kronecker delta

This notation is optional and is not examinable. The component calculations can always be written with explicit sums. An index is a compact label for a Cartesian component. Let $i = 1, 2, 3$ label the three Cartesian axes, so A_i means A_x, A_y, A_z . An index appearing twice in one term is summed over:

$$A_i B_i \equiv \sum_{i=1}^3 A_i B_i = A_x B_x + A_y B_y + A_z B_z.$$

This is the **Einstein summation convention**. A repeated index is a summation label; renaming it consistently leaves the sum unchanged. An index appearing once is a free index, and it must appear on both sides of an equation. All indices here refer to an orthonormal Cartesian basis.

The **Kronecker delta** is the component form of the identity:

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \quad \delta_{ij} B_j = B_i.$$

In $\delta_{ij} B_j$, only the term with $j = i$ contributes. Write $\hat{\mathbf{e}}_1 = \hat{\mathbf{x}}$, $\hat{\mathbf{e}}_2 = \hat{\mathbf{y}}$ and $\hat{\mathbf{e}}_3 = \hat{\mathbf{z}}$ for the basis vectors. Then

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij},$$

which says in one line that every basis vector has length one and different basis directions are perpendicular. The usual component formula for the dot product is that same statement written out. For $\mathbf{A} = A_i \hat{\mathbf{e}}_i$ and $\mathbf{B} = B_j \hat{\mathbf{e}}_j$,

$$\mathbf{A} \cdot \mathbf{B} = A_i B_j (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j) = A_i B_j \delta_{ij} = \sum_i A_i B_i.$$

Only matching components remain. A rotated orthonormal basis obeys the same dot-product table, so $\mathbf{A} \cdot \mathbf{B}$ takes the same numerical value in every such basis. Components change under a rotation; the scalar does not.

∴ Rotating the coordinate axes

Consider a new orthonormal basis (perpendicular unit vectors) rotated 45° counterclockwise:

$$\hat{\mathbf{e}}_1 = \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}}}{\sqrt{2}}, \quad \hat{\mathbf{e}}_2 = \frac{-\hat{\mathbf{x}} + \hat{\mathbf{y}}}{\sqrt{2}}.$$

The vectors $\hat{\mathbf{x}} + \hat{\mathbf{y}}$ and $-\hat{\mathbf{x}} + \hat{\mathbf{y}}$ each have magnitude $\sqrt{2}$, so dividing by $\sqrt{2}$ makes them unit vectors. We can also check that the two rotated vectors are perpendicular:

$$\begin{aligned}\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 &= \frac{1}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}}) \cdot (-\hat{\mathbf{x}} + \hat{\mathbf{y}}) = \frac{1}{2}(-1 + 0 - 0 + 1) = 0, \\ \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1 &= \frac{1}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}}) \cdot (\hat{\mathbf{x}} + \hat{\mathbf{y}}) = \frac{1}{2}(1 + 0 + 0 + 1) = 1, \\ \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_2 &= \frac{1}{2}(-\hat{\mathbf{x}} + \hat{\mathbf{y}}) \cdot (-\hat{\mathbf{x}} + \hat{\mathbf{y}}) = \frac{1}{2}(1 - 0 - 0 + 1) = 1.\end{aligned}$$

The new basis is therefore orthonormal.

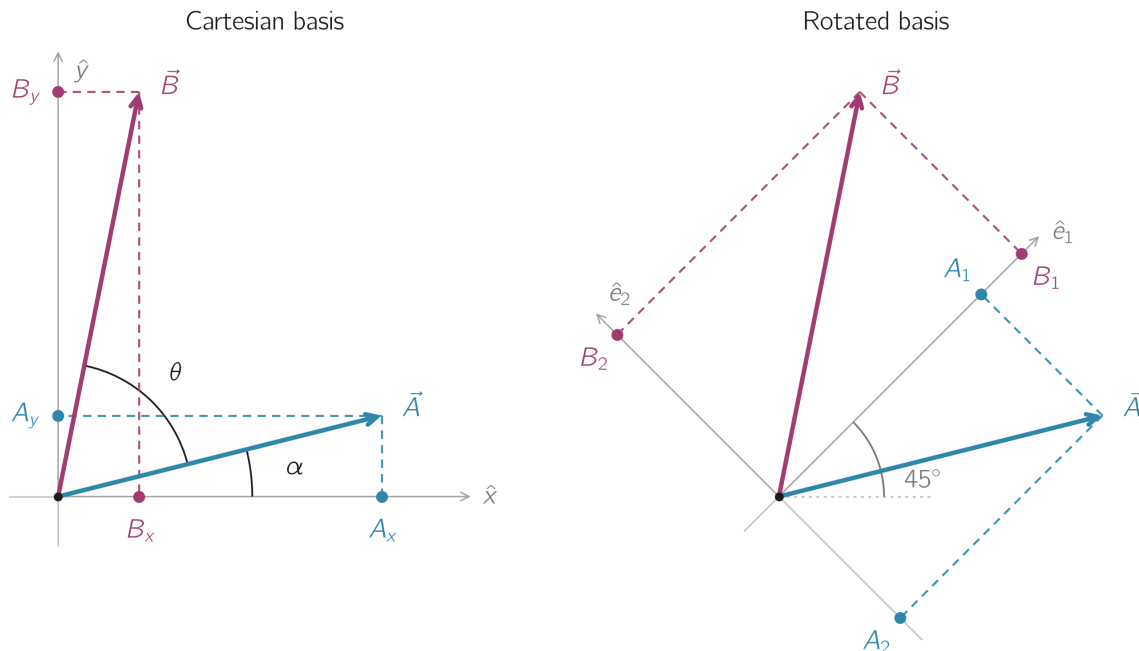
The component along a unit vector $\hat{\mathbf{e}}_i$ is $A_i = \mathbf{A} \cdot \hat{\mathbf{e}}_i$. Writing $\mathbf{A} = A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}}$, we obtain

$$\begin{aligned}A_1 &= \mathbf{A} \cdot \hat{\mathbf{e}}_1 = (A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}}) \cdot \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}}}{\sqrt{2}} \\ &= \frac{A_x(\hat{\mathbf{x}} \cdot \hat{\mathbf{x}}) + A_x(\hat{\mathbf{x}} \cdot \hat{\mathbf{y}}) + A_y(\hat{\mathbf{y}} \cdot \hat{\mathbf{x}}) + A_y(\hat{\mathbf{y}} \cdot \hat{\mathbf{y}})}{\sqrt{2}} \\ &= \frac{A_x + 0 + 0 + A_y}{\sqrt{2}} = \frac{A_x + A_y}{\sqrt{2}}, \\ A_2 &= \mathbf{A} \cdot \hat{\mathbf{e}}_2 = (A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}}) \cdot \frac{-\hat{\mathbf{x}} + \hat{\mathbf{y}}}{\sqrt{2}} \\ &= \frac{-A_x(\hat{\mathbf{x}} \cdot \hat{\mathbf{x}}) + A_x(\hat{\mathbf{x}} \cdot \hat{\mathbf{y}}) - A_y(\hat{\mathbf{y}} \cdot \hat{\mathbf{x}}) + A_y(\hat{\mathbf{y}} \cdot \hat{\mathbf{y}})}{\sqrt{2}} \\ &= \frac{-A_x + 0 - 0 + A_y}{\sqrt{2}} = \frac{-A_x + A_y}{\sqrt{2}}.\end{aligned}$$

The same vector can therefore be written as $\mathbf{A} = A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} = A_1 \hat{\mathbf{e}}_1 + A_2 \hat{\mathbf{e}}_2$. The basis has changed, so its components change as well. The same calculation holds for $\mathbf{B}$. Substituting gives

$$\begin{aligned}A_1 B_1 + A_2 B_2 &= \frac{1}{2} \left[(A_x + A_y)(B_x + B_y) + (-A_x + A_y)(-B_x + B_y) \right] \\ &= A_x B_x + A_y B_y.\end{aligned}$$

The mixed terms cancel. The components changed, but the dot product did not.



$\therefore$ Only the angle between the vectors remains

Let α be the angle from the positive x axis to $\mathbf{A}$, and let θ be the angle from $\mathbf{A}$ to $\mathbf{B}$, as drawn. Then $\mathbf{B}$ makes angle $\alpha + \theta$ with the same axis. With $A = |\mathbf{A}|$ and $B = |\mathbf{B}|$,

$$A_x = A \cos \alpha, \quad A_y = A \sin \alpha, \quad B_x = B \cos(\alpha + \theta), \quad B_y = B \sin(\alpha + \theta).$$

Multiplying matching components and adding,

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= A_x B_x + A_y B_y \\ &= AB [\cos \alpha \cos(\alpha + \theta) + \sin \alpha \sin(\alpha + \theta)] \\ &= AB \cos [(\alpha + \theta) - \alpha] \\ &= AB \cos \theta. \end{aligned}$$

The third line uses $\cos(u - v) = \cos u \cos v + \sin u \sin v$. The angle α specifying our choice of axis cancels: only the angle between the vectors remains. Rotating the basis through ϕ replaces α by $\alpha - \phi$ in both vectors' components, so the same cancellation gives $AB \cos \theta$. In three dimensions we can choose this plane to contain both vectors; their components normal to it vanish.

Σ Summing products of components

Choose any orthonormal basis $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$. The components are $A_i = \mathbf{A} \cdot \hat{\mathbf{e}}_i$ and $B_i = \mathbf{B} \cdot \hat{\mathbf{e}}_i$. Multiply the components along each matching direction, then add:

$$\mathbf{A} \cdot \mathbf{B} = \sum_{i=1}^3 A_i B_i = \sum_{i=1}^3 (\mathbf{A} \cdot \hat{\mathbf{e}}_i)(\mathbf{B} \cdot \hat{\mathbf{e}}_i).$$

This is a sum of products, not a product of sums. Expanding the vectors explains why only matching directions contribute:

$$\left(\sum_i A_i \hat{\mathbf{e}}_i \right) \cdot \left(\sum_j B_j \hat{\mathbf{e}}_j \right) = \sum_i \sum_j A_i B_j (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j) = \sum_i A_i B_i.$$

Different basis directions are perpendicular, so the terms with $i \neq j$ vanish. Rotating the basis changes the individual components, but not the full sum.

One direction alone generally gives only part of the answer. For $\mathbf{A} = \mathbf{B} = \hat{\mathbf{y}}$, the product of their x components is zero, but the complete sum is $0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 = 1$. If we choose one basis vector along $\mathbf{A}$, its other components vanish; the sum then reduces to the single term $A(B \cos \theta)$ used above.

4.1 Projection along a direction

The dot product with a unit vector gives the component along that direction.

∴ Parallel and perpendicular components

A problem often becomes simpler when we separate a vector into components along a chosen direction and perpendicular to it, as on a slope.

Take any vector $\mathbf{A}$ and any **unit** vector $\hat{\mathbf{n}}$. Split $\mathbf{A}$ into a parallel piece and a perpendicular piece. The dot product $\mathbf{A} \cdot \hat{\mathbf{n}}$ gives the scalar component along $\hat{\mathbf{n}}$. Multiplying by $\hat{\mathbf{n}}$ gives the parallel vector,

$$\mathbf{A}_{\parallel} = (\mathbf{A} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}},$$

and everything left over is the perpendicular piece,

$$\mathbf{A}_{\perp} = \mathbf{A} - \mathbf{A}_{\parallel} = \mathbf{A} - (\mathbf{A} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}}.$$

The pieces add back to $\mathbf{A}$ by construction. To check that the remaining piece is perpendicular to $\hat{\mathbf{n}}$, take their dot product:

$$\mathbf{A}_{\perp} \cdot \hat{\mathbf{n}} = \mathbf{A} \cdot \hat{\mathbf{n}} - (\mathbf{A} \cdot \hat{\mathbf{n}})(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}) = \mathbf{A} \cdot \hat{\mathbf{n}} - \mathbf{A} \cdot \hat{\mathbf{n}} = 0,$$

where the step uses $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$, which is what makes $\hat{\mathbf{n}}$ a unit vector and is exactly why the formula demands one. If the direction is handed to you as a general vector $\mathbf{B}$, use a nonzero $\mathbf{B}$ and normalize it: $\hat{\mathbf{n}} = \mathbf{B}/|\mathbf{B}|$.

The same decomposition separates motion along a string from motion across it, or radial motion from motion around a circle. The choice of direction changes with the geometry of the problem.

4.1.1 Choosing a basis along the acceleration

A useful basis can simplify the equation of motion. For a nonzero constant acceleration, define $a = |\mathbf{a}|$ and $\hat{\mathbf{a}} = \mathbf{a}/a$. Choose a fixed perpendicular unit vector $\hat{\mathbf{a}}_{\perp}$ in the plane of motion. Then

$$\mathbf{v} = v_{\parallel}\hat{\mathbf{a}} + v_{\perp}\hat{\mathbf{a}}_{\perp}, \quad \dot{\mathbf{v}} = a\hat{\mathbf{a}}.$$

Because the basis vectors are constant, this vector equation is equivalent to

$$\dot{v}_{\parallel} = a, \quad \dot{v}_{\perp} = 0.$$

Acceleration changes only the component along itself. First integrate the velocity equations:

$$v_{\parallel}(t) = v_{\parallel,0} + at, \quad v_{\perp}(t) = v_{\perp,0}.$$

Now write $\mathbf{r} = r_{\parallel}\hat{\mathbf{a}} + r_{\perp}\hat{\mathbf{a}}_{\perp}$. The equation $\ddot{\mathbf{r}} = a\hat{\mathbf{a}}$ gives

$$\ddot{r}_{\parallel} = a, \quad \ddot{r}_{\perp} = 0.$$

Integrating once more gives

$$\begin{aligned} r_{\parallel}(t) &= r_{\parallel,0} + v_{\parallel,0}t + \frac{1}{2}at^2, \\ r_{\perp}(t) &= r_{\perp,0} + v_{\perp,0}t. \end{aligned}$$

4.1.2 Velocity and acceleration at the apex

For gravity, $\hat{\mathbf{a}}$ points downward: $v_{\parallel}$ is negative on the way up, zero at the apex, and positive on the way down. Since $\mathbf{v} \cdot \mathbf{a} = av_{\parallel}$, the dot product vanishes at the apex. The perpendicular component stays constant, so

$$|\mathbf{v}| = \sqrt{v_{\parallel}^2 + v_{\perp}^2} \geq |v_{\perp}|.$$

For a projectile with nonzero horizontal velocity, its speed is smallest at the apex, where $\mathbf{v} \cdot \mathbf{a} = 0$. The velocity is not zero there. Choosing the basis changes the components, not the motion.

5 The cross product (vector product)

The dot product takes two vectors and returns a scalar. We can also define a product that returns a vector, $\mathbf{C} = \mathbf{A} \times \mathbf{B}$. To specify this new vector, we must give its direction and magnitude. This is the **cross product**, also called the **vector product**, in three-dimensional space.

For two nonparallel vectors, choose $\mathbf{C}$ perpendicular to both:

$$\mathbf{C} \cdot \mathbf{A} = 0, \quad \mathbf{C} \cdot \mathbf{B} = 0.$$

Perpendicularity alone does not specify the vector: any length along that perpendicular line would satisfy these two equations, and either sense along the line is possible. We must choose

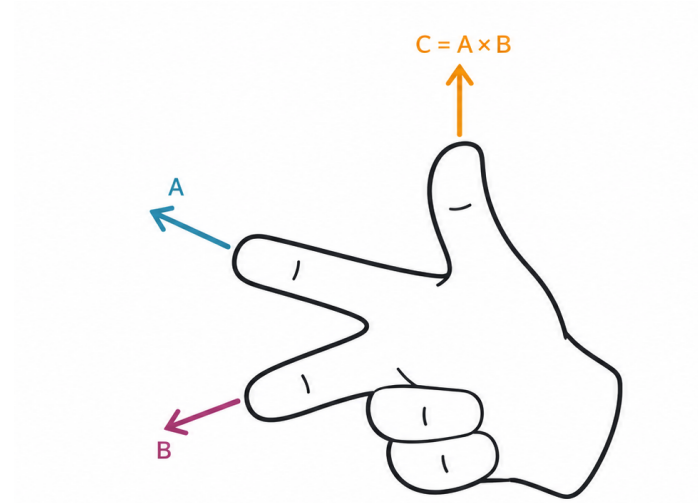


Figure 5: The right-hand rule, shown for perpendicular $\mathbf{A}$ and $\mathbf{B}$: index finger along $\mathbf{A}$, middle finger along $\mathbf{B}$, and thumb along $\mathbf{C} = \mathbf{A} \times \mathbf{B}$.

a sense and then a magnitude. The **right-hand rule** chooses the sense: curl the fingers of your right hand from $\mathbf{A}$ toward $\mathbf{B}$ through the smaller angle; your thumb points along $\mathbf{A} \times \mathbf{B}$. Reversing the order reverses this direction.

We still need a magnitude. The two vectors span a parallelogram with base $A = |\mathbf{A}|$ and perpendicular height $B \sin \theta$, where $B = |\mathbf{B}|$ and θ is the angle between them. We define the magnitude to be its area:

$$|\mathbf{A} \times \mathbf{B}| = AB \sin \theta.$$

For parallel or antiparallel vectors the area is zero, so their cross product is the zero vector. A zero input also gives zero. Perpendicularity and area are geometric properties: they do not depend on how we choose our coordinate axes.

Choose right-handed axes, so $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$, $\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$, and $\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$. Reversing the order changes the sign. The cross product distributes over addition, and crossing a basis vector with itself gives zero. Expanding in these basis vectors gives

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y) \hat{\mathbf{x}} + (A_z B_x - A_x B_z) \hat{\mathbf{y}} + (A_x B_y - A_y B_x) \hat{\mathbf{z}}.$$

For vectors in the xy -plane ($A_z = B_z = 0$), only the $\hat{\mathbf{z}}$ component survives:

$$\mathbf{A} \times \mathbf{B} = (A_x B_y - A_y B_x) \hat{\mathbf{z}}.$$

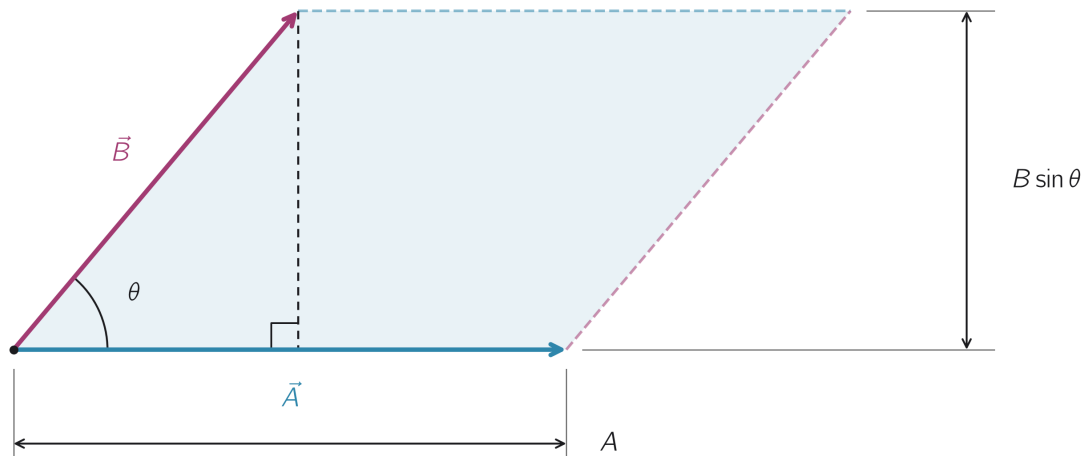


Figure 6: The parallelogram spanned by $\mathbf{A}$ and $\mathbf{B}$ has base A and perpendicular height $B \sin \theta$, so its area is $AB \sin \theta = |\mathbf{A} \times \mathbf{B}|$.

$\therefore$ Cross product for planar vectors

Distributing over the two component sums gives four terms:

$$\begin{aligned}
 \mathbf{A} \times \mathbf{B} &= (A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}}) \times (B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}}) \\
 &= A_x B_x (\hat{\mathbf{x}} \times \hat{\mathbf{x}}) + A_x B_y (\hat{\mathbf{x}} \times \hat{\mathbf{y}}) \\
 &\quad + A_y B_x (\hat{\mathbf{y}} \times \hat{\mathbf{x}}) + A_y B_y (\hat{\mathbf{y}} \times \hat{\mathbf{y}}) \\
 &= A_x B_y \hat{\mathbf{z}} - A_y B_x \hat{\mathbf{z}} \\
 &= (A_x B_y - A_y B_x) \hat{\mathbf{z}}.
 \end{aligned}$$

The first and last terms vanish because a vector crossed with itself is the zero vector. The middle two point in opposite z directions because reversing the order of a cross product changes its sign.

↗ Beyond the basics: the determinant shortcut

This shortcut is optional and is not required for this course. In a right-handed orthonormal basis, the component formula can be remembered using the formal determinant

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

Expand along the first row, with alternating signs $+$, $-$, $+$:

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{x}}(A_y B_z - A_z B_y) - \hat{\mathbf{y}}(A_x B_z - A_z B_x) + \hat{\mathbf{z}}(A_x B_y - A_y B_x).$$

For each term, delete the first row and the column of its basis vector; the remaining two-by-two determinant is the product of its main diagonal minus the product of its other diagonal. The minus sign on the $\hat{\mathbf{y}}$ term is essential. This is a mnemonic for the component formula, not an ordinary scalar determinant: the first row contains vectors.

↗ Beyond the basics: the Levi-Civita symbol

This notation is an optional mathematical extension and is not required for this course. The Levi-Civita symbol plays an analogous role for the cross product to the Kronecker delta above for the dot product. In a right-handed orthonormal basis,

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}, \quad \hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \sum_{k=1}^3 \varepsilon_{ijk} \hat{\mathbf{e}}_k.$$

The delta selects matching directions; the Levi-Civita symbol selects the perpendicular direction and its sign.

The **Levi-Civita symbol** ε_{ijk} records the signs of the right-hand rule. For $i, j, k = 1, 2, 3$,

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1, \quad \varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1,$$

and $\varepsilon_{ijk} = 0$ whenever any two indices are equal. Each index takes the values 1, 2, 3, equivalently x, y, z , in a right-handed orthonormal basis. Write $\mathbf{C} = \mathbf{A} \times \mathbf{B}$. With all sums explicit, the full vector is

$$\mathbf{C} = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} A_j B_k \hat{\mathbf{e}}_i.$$

The outer sum assembles the three components into a vector. To find one component, fix i ; only two sums remain:

$$C_i = \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} A_j B_k.$$

For example, fix $i = 3$ (the z component). The nine pairs (j, k) are $(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)$. All terms vanish except $(1, 2)$ and $(2, 1)$:

$$C_z = \varepsilon_{312} A_1 B_2 + \varepsilon_{321} A_2 B_1 = A_x B_y - A_y B_x.$$

The permutation 312 is cyclic and has sign $+1$; 321 is anticyclic and has sign -1 . Similarly, 213 has sign -1 : swapping the first two entries of 123 reverses the sign.

Using the Einstein convention introduced above, omit the sum signs over repeated indices:

$$C_i = \varepsilon_{ijk} A_j B_k.$$

Here j and k each occur twice and are summed; i occurs once on each side and is not summed. It labels which component is being calculated. The notation changes, not the calculation. In our oriented Cartesian setting, this is also called the totally antisymmetric tensor. Swapping any two indices changes the sign:

$$\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj}.$$

If two indices coincide, swapping them leaves the entry unchanged but also reverses its sign, so that entry must be zero. A cyclic permutation involves two swaps and therefore preserves the sign.

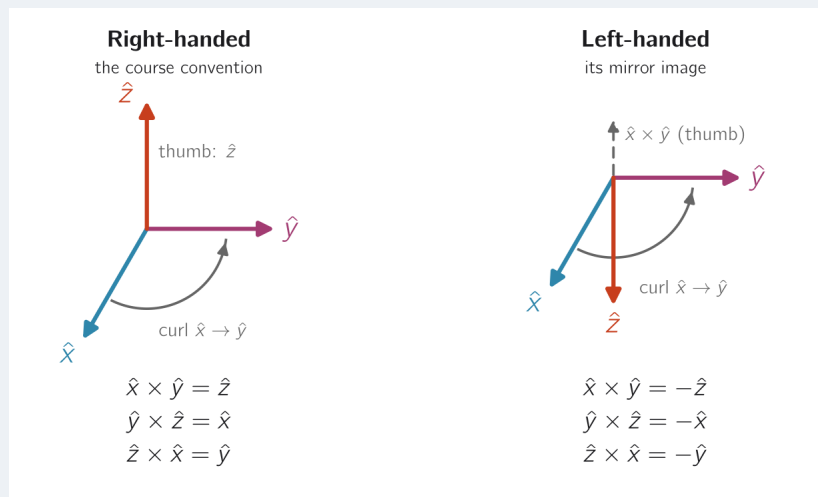
The symbol packages the same sign information as the cyclic basis products $\hat{x} \times \hat{y} = \hat{z}$ and $\hat{y} \times \hat{x} = -\hat{z}$.

In the **3D cross-product demonstration** ↗, change either vector and rotate the view to inspect the perpendicular direction. Reversing the order reverses the normal while preserving the parallelogram's area.

For vectors drawn in the xy -plane, the cross product points along the perpendicular z -axis. A dot in a circle represents an arrow pointing out of the page; a cross in a circle represents one pointing into it.

Σ Choosing an orientation in 3D space

Once $\hat{x}$ and $\hat{y}$ are fixed, either perpendicular direction can be chosen for the positive z axis. Choosing the direction given by the right-hand rule makes the ordered basis right-handed. Reversing $\hat{z}$ makes it left-handed. This is a choice of orientation, not a physical distinction between the axes.



With the usual right-hand convention for the cross product, $\hat{x} \times \hat{y} = \hat{z}$ in the left panel and $\hat{x} \times \hat{y} = -\hat{z}$ in the right panel. The cyclic basis products likewise acquire a minus sign in a left-handed basis. We use right-handed bases throughout these notes. The right-hand rule is a convention. We could instead adopt the opposite convention throughout, provided every orientation-dependent definition and sign were changed consistently; the physical predictions would be unchanged. Choosing left-handed coordinates while keeping our right-hand convention is also possible, but then the usual component cross-product formula acquires an overall minus sign.

6 Derivatives of vectors

To see the acceleration geometrically, take the velocity arrows at two nearby times. They are drawn at different points on the path, so first copy them without rotating or stretching them until their tails coincide. The arrow from the head of $\mathbf{v}(t)$ to the head of $\mathbf{v}(t + \Delta t)$ is $\Delta \mathbf{v}$. Dividing by Δt gives the average acceleration; the limit as $\Delta t \rightarrow 0$ gives $\mathbf{a}(t)$. Joining the tips of arrows left at different points on the path would not give their vector difference.

Speed is the *magnitude* of the velocity vector:

$$\text{speed} = |\mathbf{v}| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}.$$

Speed is a scalar; velocity is a vector. A car going around a roundabout at constant speed has a *changing* velocity, because the direction rotates, and therefore a nonzero acceleration.

We have already defined the derivative of a vector. For products, differentiate one factor at a time and add the terms, keeping the factors in their original order. Let $c(t)$ be a scalar and $\mathbf{A}(t)$ and $\mathbf{B}(t)$ be differentiable vectors. The product rules are

$$\begin{aligned}\frac{d}{dt}(c\mathbf{A}) &= \dot{c}\mathbf{A} + c\dot{\mathbf{A}}, \\ \frac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) &= \dot{\mathbf{A}} \cdot \mathbf{B} + \mathbf{A} \cdot \dot{\mathbf{B}}, \\ \frac{d}{dt}(\mathbf{A} \times \mathbf{B}) &= \dot{\mathbf{A}} \times \mathbf{B} + \mathbf{A} \times \dot{\mathbf{B}}.\end{aligned}$$

Keep the order of the factors in the cross product: exchanging them changes the sign. Expanding the dot product as $A_x B_x + A_y B_y + A_z B_z$ verifies its derivative rule term by term.

6.1 Changing magnitude and direction

What does the time derivative do to a vector's magnitude and direction? Wherever $\mathbf{A}(t) \neq 0$, write $\mathbf{A} = A\hat{\mathbf{n}}$, where $A = |\mathbf{A}|$ and $\hat{\mathbf{n}}$ is its unit direction. The product rule gives

$$\frac{d\mathbf{A}}{dt} = \frac{dA}{dt}\hat{\mathbf{n}} + A\frac{d\hat{\mathbf{n}}}{dt}.$$

The first term changes the magnitude; the second changes the direction. A vector can have constant magnitude and still have a nonzero derivative.

For position, this becomes

$$\mathbf{r} = r\hat{\mathbf{r}}, \quad \mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}}.$$

Here $\dot{r}$ is the radial component of velocity, not generally the speed. The velocity need not point along the position vector: it can also change the direction $\hat{\mathbf{r}}$.

For velocity, write $\mathbf{v} = v\hat{\mathbf{v}}$, with $v = |\mathbf{v}|$ and $\hat{\mathbf{v}}$ the direction of motion. Then

$$\mathbf{a} = \dot{v}\hat{\mathbf{v}} + v\dot{\hat{\mathbf{v}}}.$$

One term changes speed; the other changes the direction of motion. We will return to how the direction changes in the next class.

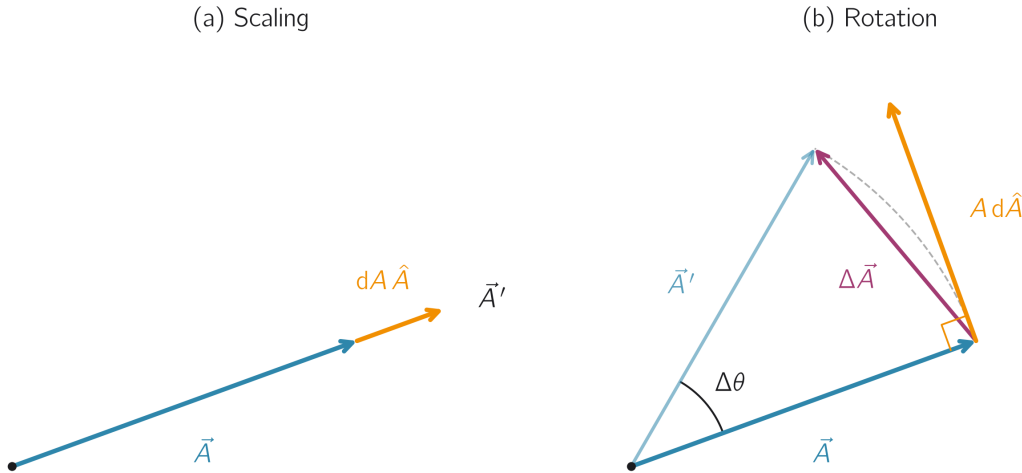


Figure 7: A vector can change through its magnitude or direction. Left: a magnitude change is parallel to the vector. Right: at fixed magnitude the tip follows a circle. The magenta arrow shows a finite change; the orange arrow schematically represents the direction-change term. We will examine this limit next class.

Summary

Basis vectors and components A basis is a set of linearly independent vectors that spans the space: every vector has a unique expansion in it. We use orthonormal bases, whose vectors have unit length and are mutually perpendicular. In three dimensions,

$$\hat{e}_i \cdot \hat{e}_j = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad i, j = 1, 2, 3.$$

The component along a basis vector is obtained by taking the inner product with it:

$$A_i = \mathbf{A} \cdot \hat{e}_i, \quad \mathbf{A} = \sum_{i=1}^3 A_i \hat{e}_i = \sum_{i=1}^3 (\mathbf{A} \cdot \hat{e}_i) \hat{e}_i.$$

Here A_i is a scalar; $A_i \hat{e}_i$ is the corresponding vector projection. A general basis need not be orthonormal; these component-extraction formulas use orthonormality. For cross products, we also choose a right-handed basis.

Vector kinematics in a fixed Cartesian basis

$$\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}, \quad \mathbf{v} = \dot{\mathbf{r}}, \quad \mathbf{a} = \ddot{\mathbf{r}}.$$

Position, velocity and acceleration have units m, m/s and m/s². Speed is $|\mathbf{v}|$, the non-negative rate of change of path length.

Vector products

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z = |\mathbf{A}| |\mathbf{B}| \cos \theta.$$

The dot product is a scalar. The cross product is perpendicular to both vectors, with magnitude $|\mathbf{A}||\mathbf{B}|\sin\theta$ and direction given by the right-hand rule. Both products have units equal to the product of their inputs' units. For vectors in the xy -plane,

$$\mathbf{A} \times \mathbf{B} = (A_x B_y - A_y B_x) \hat{\mathbf{z}}.$$

Changing speed and direction

$$\mathbf{v} = v \hat{\mathbf{v}}, \quad \mathbf{a} = \dot{v} \hat{\mathbf{v}} + v \frac{d\hat{\mathbf{v}}}{dt}.$$

The first term changes speed; the second changes direction. What determines this derivative along a curved trajectory? We return to this question next class.