

Circular Motion and Polar Coordinates

Week 03

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Readings

1. **Kleppner & Kolenkow, §§1.10–1.11.** *More about the Time Derivative of a Vector and Motion in Plane Polar Coordinates.* Derivatives of rotating vectors and the polar velocity and acceleration formulas.
2. **Feynman Lectures I, Ch. 9, §§9-2–9-3.** *Speed and velocity and Components of velocity, acceleration, and force.* [feynmanlectures.caltech.edu/I_09.html](https://www.feynmanlectures.caltech.edu/I_09.html) ↗
3. **Cole, §4.7.** *Acceleration for a particle moving in a circle.*

Overview

Last week we wrote a vector as its magnitude times its direction, $\mathbf{A} = A\hat{\mathbf{A}}$. Differentiating that product gives two contributions, one from changing the length and one from turning the direction. We left a question open: how do we calculate the change in direction?

We begin by watching the tip of a vector as it turns: it traces a circular arc, and connecting the distance along that arc to the angle turned gives the rate at which a direction changes. From this we build a basis that rotates with the particle, work through uniform circular motion explicitly, and find its inward acceleration and what happens when the circular path is released. Finally we let both the distance and the angle change arbitrarily, obtain the velocity and acceleration in polar coordinates, and follow a bug walking outward on a turning table.

1 Cartesian Coordinates and Time Derivatives

Last week we wrote a position vector in 2D as

$$\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}}.$$

To get the velocity, we differentiated with respect to time:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{x}\hat{\mathbf{x}} + \dot{y}\hat{\mathbf{y}}.$$

The dot is Newton's shorthand for a time derivative. As in Week 1, $\dot{x} \equiv dx/dt$ and $\ddot{x} \equiv d^2x/dt^2$. What did we assume? That $x(t)$ and $y(t)$ are differentiable, and one more thing. The derivative is simple because $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are *fixed* unit vectors. They point in the same direction at every instant. Their time derivatives are zero:

$$\frac{d\hat{\mathbf{x}}}{dt} = \mathbf{0}, \quad \frac{d\hat{\mathbf{y}}}{dt} = \mathbf{0}.$$

The basis vectors stay fixed. Only the components $x(t)$ and $y(t)$ change.

Today we choose a basis that *moves with the particle*. The price is that we must track how the basis vectors rotate.

△ Differentiating a moving basis

When the basis moves with the particle, its vectors can change direction. A time derivative therefore acts on both the scalar components and the basis vectors. Keeping both parts of the product rule accounts for changes in length and direction.

2 Two Ways a Vector Can Change

The position vector $\mathbf{r}$ has a magnitude $|\mathbf{r}| = r$ and a direction $\hat{\mathbf{r}} \equiv \mathbf{r}/r$. Away from the origin ($r > 0$), we can write

$$\mathbf{r}(t) = r(t) \hat{\mathbf{r}}(t).$$

This is just $\mathbf{r} = (\text{length}) \times (\text{direction})$. It separates two changes:

- The length $r(t)$ can change. The particle moves toward or away from the origin.
- The direction $\hat{\mathbf{r}}(t)$ can change. The particle swings around the origin.

Either change makes $\mathbf{r}$ a different vector at the next instant. Either change therefore contributes to the velocity.

Differentiate with the product rule:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \underbrace{\dot{r} \hat{\mathbf{r}}}_{\text{length-changing}} + \underbrace{r \dot{\hat{\mathbf{r}}}}_{\text{direction-changing}}.$$

The first term is parallel to $\hat{\mathbf{r}}$. It tells us how fast the particle moves radially outward. The particle moves inward if $\dot{r} < 0$. The second term contains $\dot{\hat{\mathbf{r}}}$, the time derivative of a *unit vector*. Next we need to know what $\dot{\hat{\mathbf{r}}}$ looks like.

The change $d\hat{\mathbf{r}} = \hat{\mathbf{r}}(t+dt) - \hat{\mathbf{r}}(t)$ is the difference of two unit vectors that point in nearly the same direction, so it is small compared with $\hat{\mathbf{r}}$. Dividing by dt gives the derivative $\dot{\hat{\mathbf{r}}}$, whose magnitude is set by how fast the direction turns, not by the unit length of $\hat{\mathbf{r}}$.

To isolate the change in direction, hold the length r fixed. Then

$$d\mathbf{r} = r d\hat{\mathbf{r}}, \quad \frac{d\mathbf{r}}{dt} = r \frac{d\hat{\mathbf{r}}}{dt}.$$

The first expression is a small displacement; the second is its rate of change. The tip of the vector moves along a circle, and the drawing suggests that for a small turn the displacement points along the tangent to that circle, perpendicular to $\hat{r}$. A finite difference between two directions is a chord; its direction approaches the tangent as the turn shrinks. If this holds, changing length contributes along $\hat{r}$ while changing direction contributes perpendicular to it.

(a) length changes

(b) direction changes

(c) both change

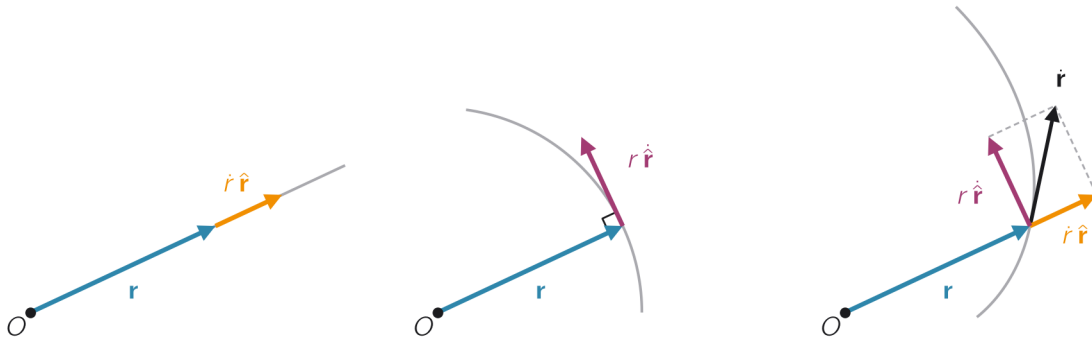


Figure 1: The two ways a position vector can change. Left: its length changes while its direction stays fixed. Middle: its direction changes while its length stays fixed; the tip follows a circular arc. Right: length and direction change together, tracing an outward spiral. Orange arrows show the radial contribution $\dot{r} \hat{r}$, and magenta arrows show the turning contribution $r \dot{\hat{r}}$. Their sum, shown in black, is the velocity tangent to the path. Position and velocity arrows use separate scales.

2.1 Relating change in direction to change in angle

The derivative $\dot{\hat{r}}$ is itself a vector. What are its direction and magnitude? Since $\hat{r}$ has fixed length 1, its tip moves along a unit circle, which suggests that the derivative is tangent to that circle; Section 2.2 proves this. Here we find its magnitude by relating the distance the tip travels to the angle turned. Let θ measure this direction from a fixed reference, increasing counterclockwise.

A small turn changes the direction of the vector. To find the size of this change, follow its tip around the circle. This is where arc length enters.

At fixed radius r , the angle in radians is the arc length divided by the radius, $\theta = s/r$. A full turn has $s = 2\pi r$ and $\theta = 2\pi$. For an infinitesimal positive turn,

$$ds = r d\theta.$$

The angle is dimensionless: it is a ratio of two lengths.

Now apply this geometry to $\hat{r}$, whose tip lies on a circle of radius 1. The straight arrow joining two nearby tips is a chord; as the turn shrinks, its length agrees with the arc to first order. Thus, in differential notation,

$$|d\hat{r}| = |d\theta|.$$

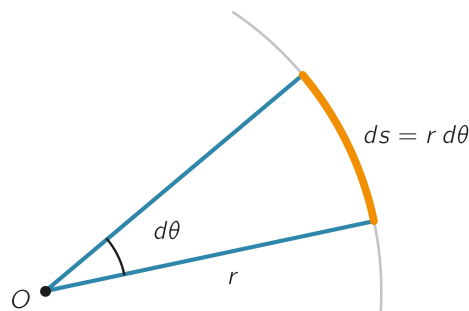


Figure 2: Schematic infinitesimal turn. Two radii of length r differ by a positive angle $d\theta$. The orange circular arc has length $ds = r d\theta$. The angle is exaggerated for visibility.

The absolute values include either sense of rotation. We use $d\theta$ and $d\hat{\mathbf{r}}$ for infinitesimal changes, with the limit already understood.

This relates the change of $\hat{\mathbf{r}}$ to a change of angle. When the vector turns with time, $\hat{\mathbf{r}}$ changes because $\theta(t)$ changes, and we convert a change per angle into a change per time with the chain rule.

Σ Chain rule

If $\mathbf{A}$ depends on time through $\theta(t)$, its differential change is

$$d\mathbf{A} = \frac{d\mathbf{A}}{d\theta} d\theta, \quad d\theta = \frac{d\theta}{dt} dt.$$

Substitute the second relation into the first and divide by dt . In differential notation, the intermediate $d\theta$ factors cancel:

$$\frac{d\mathbf{A}}{d\theta} \frac{d\theta}{dt} = \frac{d\mathbf{A}}{dt}.$$

Change per angle, multiplied by angle per time, gives change per time. The differentials already include the infinitesimal limit.

To describe the turn in time, define $\dot{\theta} = d\theta/dt$. On the fixed-radius circle shown, $\dot{s} = r\dot{\theta}$ for counterclockwise motion; for either sense, the distance traveled per time is $r|\dot{\theta}|$. For the unit vector, $|d\hat{\mathbf{r}}| = |d\theta|$ and the chain rule give

$$\left| \frac{d\hat{\mathbf{r}}}{dt} \right| = |\dot{\theta}|.$$

The magnitude of the rate of change of direction is the absolute angular rate. Next we find its direction exactly.

2.2 The direction of the changing unit vector

The picture suggests that $d\hat{\mathbf{r}}/dt$ is perpendicular to $\hat{\mathbf{r}}$. Is this exact? It follows directly from the fact that $\hat{\mathbf{r}}$ is a unit vector: $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1$. Differentiating with the dot-product rule from last week gives

$$0 = \frac{d}{dt}(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}) = 2\hat{\mathbf{r}} \cdot \frac{d\hat{\mathbf{r}}}{dt}.$$

Thus $d\hat{\mathbf{r}}/dt$ has zero component along $\hat{\mathbf{r}}$. Whenever the derivative is nonzero, it points perpendicular to $\hat{\mathbf{r}}$; if the direction is momentarily unchanged, the derivative is zero. For motion in a fixed plane, it also lies in that plane and is therefore perpendicular to its unit normal $\hat{\mathbf{n}}$.

The argument uses only that the length of $\hat{\mathbf{r}}$ does not change. For any vector with constant magnitude, differentiating $\mathbf{A} \cdot \mathbf{A} = \text{const}$ gives

$$\mathbf{A} \cdot \frac{d\mathbf{A}}{dt} = 0.$$

A change that preserves the length of a vector is a rotation, and its rate of change is perpendicular to the vector.

Choose the sense of $\hat{\mathbf{n}}$ so that increasing θ is counterclockwise when viewed from its tip. Which vector is perpendicular to both $\hat{\mathbf{n}}$ and $\hat{\mathbf{r}}$? The cross product we learned last week supplies it:

$$\hat{\boldsymbol{\theta}} \equiv \hat{\mathbf{n}} \times \hat{\mathbf{r}}, \quad \hat{\boldsymbol{\theta}} \cdot \hat{\mathbf{r}} = 0, \quad \hat{\boldsymbol{\theta}} \cdot \hat{\mathbf{n}} = 0.$$

Because $\hat{\mathbf{n}}$ and $\hat{\mathbf{r}}$ are perpendicular unit vectors, their cross product has magnitude 1. The right-hand rule selects the sense of increasing θ . This is the tangent direction suggested by the geometry; we now name it $\hat{\boldsymbol{\theta}}$.

Tangent to what? The vector $\hat{\boldsymbol{\theta}}$ is tangent to the circle *centered at the origin* that passes through the particle's current position. It is fixed by the position alone. It need not be tangent to the path the particle actually follows; that role belongs to the full velocity, as we will see in Section 5.

The arc-length argument gives $|\dot{\hat{\mathbf{r}}}| = |\dot{\theta}|$, and the perpendicularity argument fixes the direction. When θ increases, the change points along $\hat{\boldsymbol{\theta}}$; when it decreases, the change points opposite to $\hat{\boldsymbol{\theta}}$. Together,

$$\boxed{\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta} \hat{\boldsymbol{\theta}}}.$$

5 Return to the product rule for $\mathbf{r} = r\hat{\mathbf{r}}$. For an infinitesimal change,

$$d\mathbf{r} = dr \hat{\mathbf{r}} + r d\hat{\mathbf{r}} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}}.$$

Here $d\hat{\mathbf{r}} = d\theta \hat{\boldsymbol{\theta}}$ follows by multiplying the time-derivative equation by dt . Both terms have units of length: $dr \hat{\mathbf{r}}$ is the radial displacement, and $r d\theta \hat{\boldsymbol{\theta}}$ is the sideways displacement from turning.

Σ Alternative derivation: the chain rule

One can instead differentiate with respect to the angle. On the unit circle, $|d\hat{\mathbf{r}}| = |d\theta|$, and an increase in θ changes $\hat{\mathbf{r}}$ along $+\hat{\boldsymbol{\theta}}$. Therefore

$$\frac{d\hat{\mathbf{r}}}{d\theta} = \hat{\boldsymbol{\theta}}.$$

The chain rule then gives the same time derivative:

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\hat{\mathbf{r}}}{d\theta} \frac{d\theta}{dt} = \dot{\theta} \hat{\boldsymbol{\theta}}.$$

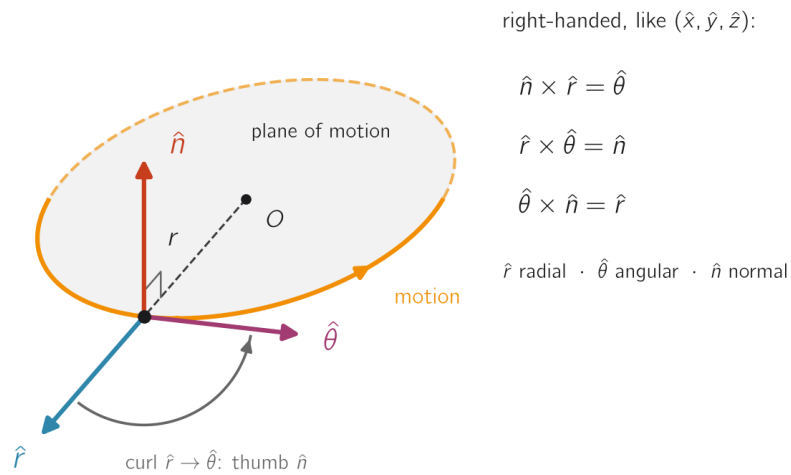


Figure 3: The rotating triad on a circular path. Here $\hat{\mathbf{n}}$ is the unit normal to the shaded plane of motion, chosen by the right-hand rule: curl from $\hat{\mathbf{r}}$ toward increasing θ and $\hat{\mathbf{n}}$ is your thumb. The motion shown has $\dot{\theta} > 0$. At the particle, $\hat{\mathbf{r}}$ points radially outward and $\hat{\boldsymbol{\theta}} = \hat{\mathbf{n}} \times \hat{\mathbf{r}}$ lies in the plane, along the motion. The products of $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{n}})$ cycle like those of $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$, but $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ ride around with the particle while $\hat{\mathbf{n}}$ never moves.

What are the units of these vectors? The three unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\mathbf{n}}$ are dimensionless. The derivative $d\hat{\mathbf{r}}/dt$ has units of inverse time; it is not itself a unit vector. The angular rate supplies those units in $d\hat{\mathbf{r}}/dt = \dot{\theta} \hat{\boldsymbol{\theta}}$.

The normal is constant, $d\hat{\mathbf{n}}/dt = \mathbf{0}$, only because the plane of motion is fixed; if the plane tilts, $\hat{\mathbf{n}}$ turns as well.

Together, $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{n}})$ form a right-handed orthonormal basis in three dimensions. The first two directions lie in the plane of motion; the third is normal to it. Their cross products cycle in this order:

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{n}}, \quad \hat{\boldsymbol{\theta}} \times \hat{\mathbf{n}} = \hat{\mathbf{r}}, \quad \hat{\mathbf{n}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}.$$

Reversing either pair changes the sign.

Σ Clockwise and counterclockwise

The positive angular direction is a convention, like the positive direction of an axis. Choose the normal up or down; the ordered triad $(\hat{r}, \hat{\theta}, \hat{n})$ remains right-handed with $\hat{\theta} = \hat{n} \times \hat{r}$. Reversing the normal therefore reverses $\hat{\theta}$ too:

$$(\hat{r}, \hat{\theta}, \hat{n}) \longrightarrow (\hat{r}, -\hat{\theta}, -\hat{n}).$$

With $\hat{n}$ toward the viewer, counterclockwise is positive and clockwise motion has $\dot{\theta} < 0$. Reversing the particle's motion does not change this convention. Reversing the convention for the same motion flips both $\dot{\theta}$ and $\hat{\theta}$, leaving $\dot{\theta} \hat{\theta}$ unchanged.

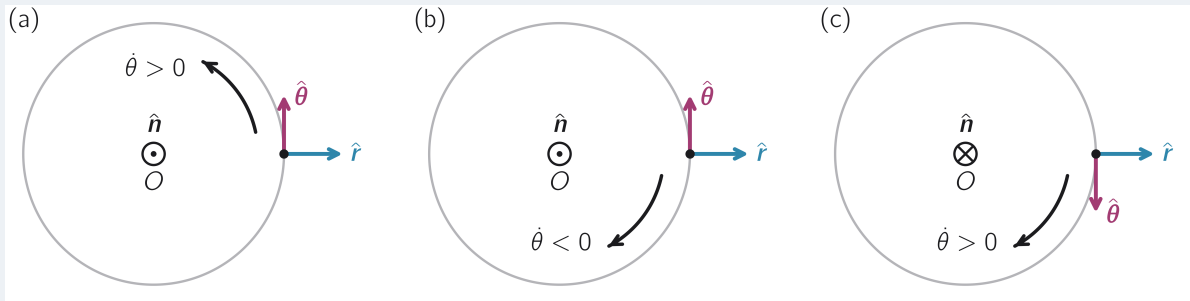


Figure 4: (a) With $\hat{n}$ toward the viewer ($\odot$), counterclockwise motion has $\dot{\theta} > 0$. (b) The same basis, with clockwise motion: $\dot{\theta} < 0$. (c) The same clockwise motion as in (b), with the convention reversed: both $\hat{n}$ and $\hat{\theta}$ reverse, and $\dot{\theta} > 0$. The product $\dot{\theta} \hat{\theta}$ is unchanged.

Any vector can be resolved along these three directions, with components obtained by dot products:

$$\mathbf{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_n \hat{n}, \quad A_r = \mathbf{A} \cdot \hat{r}, \quad A_\theta = \mathbf{A} \cdot \hat{\theta}, \quad A_n = \mathbf{A} \cdot \hat{n}.$$

Guess first

Now let us decompose the position vector in this basis, with the origin in the plane of motion:

$$\mathbf{r} = r_r \hat{r} + r_\theta \hat{\theta} + r_n \hat{n}.$$

Three possible components. What are they?

Before calculating, look at how we chose $\hat{r}$: it points along the position vector itself. Taking the dot products confirms it:

$$r_r = \mathbf{r} \cdot \hat{r} = r, \quad r_\theta = \mathbf{r} \cdot \hat{\theta} = 0, \quad r_n = \mathbf{r} \cdot \hat{n} = 0.$$

Only one component survives:

$$\mathbf{r} = r \hat{r}.$$

The angular component is zero even when the particle is turning. The angle changes the direction of the basis vector $\hat{r}$; it does not add a second component to position. That is the advantage of choosing a basis that follows the particle.

This is a basis attached to the point: $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ rotate as its angular position changes, while the normal to the fixed plane stays constant. How can we use this construction to specify the position of the particle?

3 Polar Coordinates and Circular Motion

This construction suggests a different way to specify position: give the distance from the origin and the direction. We have arrived at the polar coordinate system. Cartesian coordinates are (x, y, z) , with basis $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$. In the plane of motion, polar coordinates are (r, θ) , with in-plane basis $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}})$.

The pair $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}})$ is called the polar basis. Completing it with the normal $\hat{\mathbf{n}}$ gives the cylindrical basis $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{n}})$ in three dimensions. For the two-dimensional motion considered here, the particle remains in a fixed plane through the origin, perpendicular to the fixed unit normal $\hat{\mathbf{n}}$. How do we find the transformation between the bases? Start with the full Cartesian expression for position:

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}.$$

Choose the Cartesian axes so the motion lies in the xy plane. Then $z = 0$, and the same position vector is $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} = r\hat{\mathbf{r}}$. Taking dot products with the Cartesian basis vectors gives the coordinate relations

$$x = \mathbf{r} \cdot \hat{\mathbf{x}} = r \cos \theta, \quad y = \mathbf{r} \cdot \hat{\mathbf{y}} = r \sin \theta.$$

Dividing by $r > 0$ gives $\hat{\mathbf{r}} \cdot \hat{\mathbf{x}} = \cos \theta$ and $\hat{\mathbf{r}} \cdot \hat{\mathbf{y}} = \sin \theta$. These two components together specify the direction.

The basis constructed in Section 2 now has a coordinate interpretation: $\hat{\mathbf{r}}$ points toward increasing r , and $\hat{\boldsymbol{\theta}}$ toward increasing θ . The figure shows how these directions change from point to point while the Cartesian basis stays fixed.

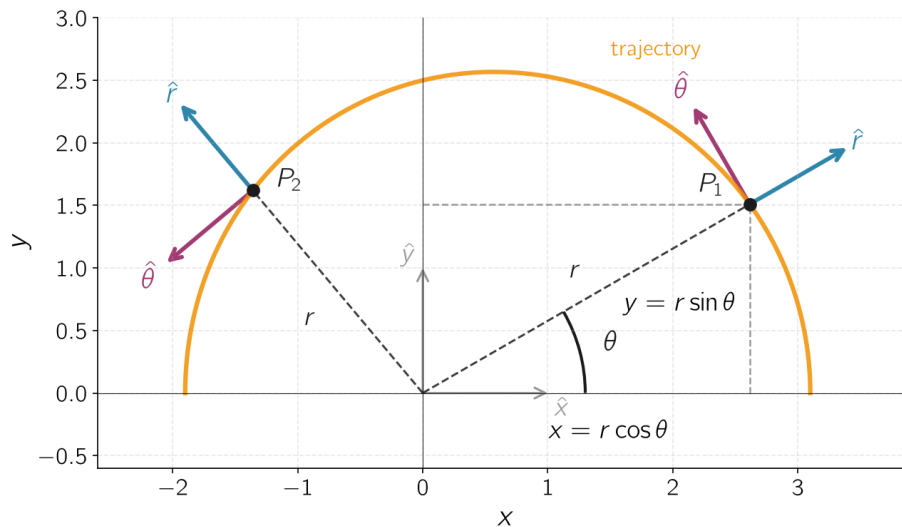


Figure 5: Polar unit vectors rotate with the particle. The basis $(\hat{r}, \hat{\theta})$ is drawn at two points P_1 and P_2 of one trajectory. Both vectors have length 1 at every instant, but they point in different directions as θ changes. The Cartesian basis $(\hat{x}, \hat{y})$, in gray at the origin, stays put. The dashed projections at P_1 show $x = r \cos \theta$ and $y = r \sin \theta$.

Written in Cartesian components, the polar basis is

$$\hat{r} = \cos \theta \hat{x} + \sin \theta \hat{y}, \quad \hat{\theta} = -\sin \theta \hat{x} + \cos \theta \hat{y}.$$

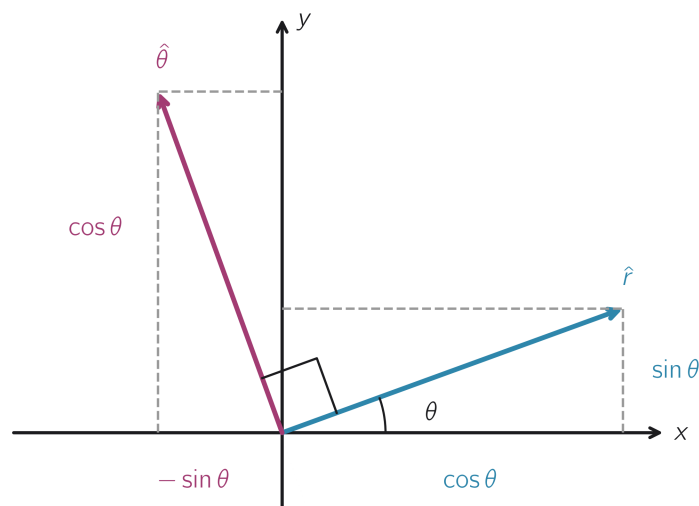


Figure 6: The perpendicular unit vectors on the same Cartesian axes: $\hat{r} = \cos \theta \hat{x} + \sin \theta \hat{y}$ and $\hat{\theta} = -\sin \theta \hat{x} + \cos \theta \hat{y}$. The dashed guides show their projections; the horizontal projection of $\hat{\theta}$ points toward $-\hat{x}$.

These equations express the rotating basis vectors in the fixed Cartesian basis. The vector $\hat{\mathbf{r}}$ makes angle θ with $\hat{\mathbf{x}}$, and $\hat{\boldsymbol{\theta}}$ is perpendicular to it in the direction of increasing θ . Its horizontal projection points toward $-\hat{\mathbf{x}}$, which accounts for the minus sign. The normal $\hat{\mathbf{n}}$ stays fixed.

As a check, $\hat{\boldsymbol{\theta}}$ is $\hat{\mathbf{r}}$ turned by 90° : $\hat{\boldsymbol{\theta}}(\theta) = \hat{\mathbf{r}}(\theta + \pi/2)$. Since $\cos(\theta + \pi/2) = -\sin \theta$ and $\sin(\theta + \pi/2) = \cos \theta$, this reproduces both components.

For any vector in the plane, $\mathbf{A} = A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}}$, the polar components follow from one dot product each, the same procedure as in the Cartesian basis:

$$A_r = \mathbf{A} \cdot \hat{\mathbf{r}} = A_x \cos \theta + A_y \sin \theta, \quad A_\theta = \mathbf{A} \cdot \hat{\boldsymbol{\theta}} = -A_x \sin \theta + A_y \cos \theta.$$

The components are scalars. They depend on the point, through θ , at which the polar basis is attached. The length does not: $A_r^2 + A_\theta^2 = A_x^2 + A_y^2$.

Recall the distinction from Section 2.2: (r, θ) specify the point, whereas $(r, 0)$ are the components of its position vector in the polar basis. The angle determines the basis direction.

3.1 Uniform circular motion

Consider a particle whose Cartesian coordinates are

$$x(t) = R \cos(\Omega t), \quad y(t) = R \sin(\Omega t),$$

with constants $R > 0$ and $\Omega > 0$. Here R and Ω are fixed parameters. R has units of length and Ω is a scalar with units of inverse time, so the argument Ωt is a dimensionless angle. Because x and y are $\cos(\Omega t)$ and $\sin(\Omega t)$ multiplied by the same positive constant, the position makes the angle Ωt with the positive x axis. This example is the choice

$$\theta(t) = \Omega t.$$

Write the position vector and factor out R :

$$\mathbf{r}(t) = R \underbrace{[\cos(\Omega t) \hat{\mathbf{x}} + \sin(\Omega t) \hat{\mathbf{y}}]}_{\hat{\mathbf{r}}(t)} = R \hat{\mathbf{r}}(t).$$

The bracket is the radial unit vector of Section 3 evaluated at $\theta = \Omega t$. How far is the particle from the origin? The length follows from the dot product, $|\mathbf{r}|^2 = \mathbf{r} \cdot \mathbf{r} = x^2 + y^2$:

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = R^2 [\cos^2(\Omega t) + \sin^2(\Omega t)] = R^2,$$

so

$$r(t) = |\mathbf{r}(t)| = R, \quad \dot{r} = 0.$$

The distance from the origin is a constant of motion: the position changes direction, but its magnitude does not change with time. The particle moves on a circle of radius R in a fixed plane.

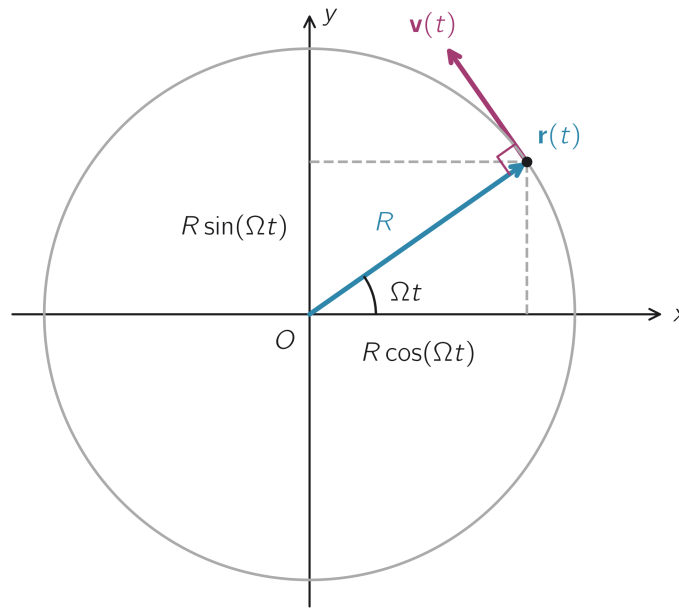


Figure 7: Uniform circular motion: the position vector $\mathbf{r}(t)$ has fixed magnitude R and angle Ωt from the positive x axis. Its Cartesian projections are $R \cos(\Omega t)$ and $R \sin(\Omega t)$. The magenta velocity arrow is tangent to the circle, perpendicular to the radius. Position and velocity arrows use separate scales.

Now differentiate the position in the fixed Cartesian basis. The chain rule brings down a factor of Ω :

$$\mathbf{v}(t) = R\Omega \underbrace{\left[-\sin(\Omega t) \hat{\mathbf{x}} + \cos(\Omega t) \hat{\mathbf{y}} \right]}_{\hat{\boldsymbol{\theta}}(t)} = R\Omega \hat{\boldsymbol{\theta}}(t).$$

The position and velocity therefore take the form

$$\begin{aligned} \mathbf{r}(t) &= R \hat{\mathbf{r}}(t), \\ \mathbf{v}(t) &= R\Omega \hat{\boldsymbol{\theta}}(t). \end{aligned}$$

In these expressions, all the time dependence is carried by the basis directions $\hat{\mathbf{r}}(t)$ and $\hat{\boldsymbol{\theta}}(t)$; the coefficients R and $R\Omega$ are constant.

The directions of the polar basis vectors change with time as the particle moves around the circle. Both $\hat{\mathbf{r}}(t)$ and $\hat{\boldsymbol{\theta}}(t)$ rotate, even though they remain unit vectors and perpendicular to each other. The Cartesian directions $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ stay fixed. Thus the velocity changes direction through $\hat{\boldsymbol{\theta}}(t)$ even though its prefactor $R\Omega$ is constant. This time dependence of the basis must be included when we differentiate.

We can check the perpendicularity explicitly from the Cartesian expressions:

$$\mathbf{r}(t) \cdot \mathbf{v}(t) = R^2\Omega \left[-\cos(\Omega t) \sin(\Omega t) + \sin(\Omega t) \cos(\Omega t) \right] = 0.$$

Position and velocity are perpendicular at every instant. Using $\hat{\boldsymbol{\theta}}(t) = \hat{\mathbf{n}} \times \hat{\mathbf{r}}(t)$, we can also write $\mathbf{v} = R\Omega(\hat{\mathbf{n}} \times \hat{\mathbf{r}})$. At $t = 0$, the particle is on the positive x axis and its velocity points along $\hat{\mathbf{y}}$; when $\Omega t = \pi/2$, it is on the positive y axis and its velocity points along $-\hat{\mathbf{x}}$. The cross product gives both directions with the same rule.

Guess first

The velocity changes with time. Does its magnitude change too?

Taking the dot product of the velocity with itself,

$$\mathbf{v}(t) \cdot \mathbf{v}(t) = R^2\Omega^2 [\sin^2(\Omega t) + \cos^2(\Omega t)] = R^2\Omega^2,$$

so $v = |\mathbf{v}| = R\Omega$. The speed does not depend on time: it is a second constant of motion. The velocity vector itself changes direction.

✓ Units — Velocity

R has units of length, Ω has units of inverse time, and the unit vector $\hat{\boldsymbol{\theta}}(t)$ is dimensionless. Therefore $R\Omega\hat{\boldsymbol{\theta}}(t)$ has units of length per time, as a velocity must. In SI units, $\text{m} \times \text{s}^{-1} = \text{m/s}$. The trigonometric factors are dimensionless because their argument Ωt is dimensionless.

The same velocity follows from Section 2 without Cartesian components. With $\mathbf{r} = R\hat{\mathbf{r}}$, the product rule gives

$$\mathbf{v} = \frac{d}{dt}(R\hat{\mathbf{r}}) = \dot{R}\hat{\mathbf{r}} + R\dot{\hat{\mathbf{r}}} = R\dot{\hat{\mathbf{r}}} = R\dot{\theta}\hat{\boldsymbol{\theta}},$$

since $\dot{R} = 0$ and $\dot{\hat{\mathbf{r}}} = \dot{\theta}\hat{\boldsymbol{\theta}}$. Here $\dot{\theta} = \Omega$, so this is $R\Omega\hat{\boldsymbol{\theta}}$, matching the Cartesian result.

When does the motion repeat? A period T must satisfy $\mathbf{r}(t + T) = \mathbf{r}(t)$ at every t . The Cartesian expressions require

$$\begin{aligned}\cos(\Omega t + \Omega T) &= \cos(\Omega t), \\ \sin(\Omega t + \Omega T) &= \sin(\Omega t).\end{aligned}$$

Using the angle-addition formulas,

$$\begin{aligned}\cos(\Omega t + \Omega T) &= \cos(\Omega t)\cos(\Omega T) - \sin(\Omega t)\sin(\Omega T), \\ \sin(\Omega t + \Omega T) &= \sin(\Omega t)\cos(\Omega T) + \cos(\Omega t)\sin(\Omega T).\end{aligned}$$

Both coordinates repeat when $\cos(\Omega T) = 1$ and $\sin(\Omega T) = 0$: $\Omega T = 2\pi k$ for an integer k . The smallest positive return time, for $\Omega > 0$, is

$$T = \frac{2\pi}{\Omega}, \quad \Omega = \frac{2\pi}{T}.$$

The period has units of time and Ω has units of inverse time. Their product is the dimensionless angle of one full turn.

Σ Angle-addition formulas from exponentials (optional)

You can derive these identities, or simply remember the formulas if you prefer. Euler's formula, with $i^2 = -1$, is

$$e^{i\alpha} = \cos \alpha + i \sin \alpha.$$

Multiply two such expressions and use $e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}$:

$$\begin{aligned} e^{i(\alpha+\beta)} &= (\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) \\ &= (\cos \alpha \cos \beta - \sin \alpha \sin \beta) \\ &\quad + i(\sin \alpha \cos \beta + \cos \alpha \sin \beta). \end{aligned}$$

Matching real and imaginary parts gives

$$\begin{aligned} \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta, \\ \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta. \end{aligned}$$

For the period, $e^{i\Omega(t+T)} = e^{i\Omega t} e^{i\Omega T}$ returns to $e^{i\Omega t}$ when $e^{i\Omega T} = 1$.

Δ Frequency convention

We often call Ω simply frequency. Frequency in cycles per second is $f = \Omega/(2\pi) = 1/T$, measured in hertz ($1 \text{ Hz} = 1 \text{ s}^{-1}$).

In polar coordinates, the whole motion is summarized by two statements:

$$\boxed{\dot{r} = 0, \quad \dot{\theta} = \Omega, \quad \Omega = \text{constant}.}$$

Circular is the first: the distance from the center is fixed. *Uniform* is the second: the angle advances at a constant rate, so the speed $R\Omega$ is constant. A fixed radius alone leaves $\theta(t)$ free: in circular motion that is not uniform, the angular rate $\dot{\theta}(t)$ varies with time and so does the speed $R|\dot{\theta}|$.

4 Derivatives of the Polar Unit Vectors

For uniform circular motion we found

$$\mathbf{v}(t) = R\Omega\hat{\boldsymbol{\theta}}(t).$$

✍ Guess first

The speed is constant. Is the acceleration zero?

Since R and Ω are constant,

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = R\Omega \frac{d\hat{\boldsymbol{\theta}}}{dt}.$$

We know that $d\hat{r}/dt = \dot{\theta}\hat{\theta}$. What is the direction of $d\hat{\theta}/dt$? Follow the tip of $\hat{\theta}$ as the basis turns counterclockwise. Its change points toward $-\hat{r}$, just as the change of $\hat{r}$ points toward $+\hat{\theta}$.

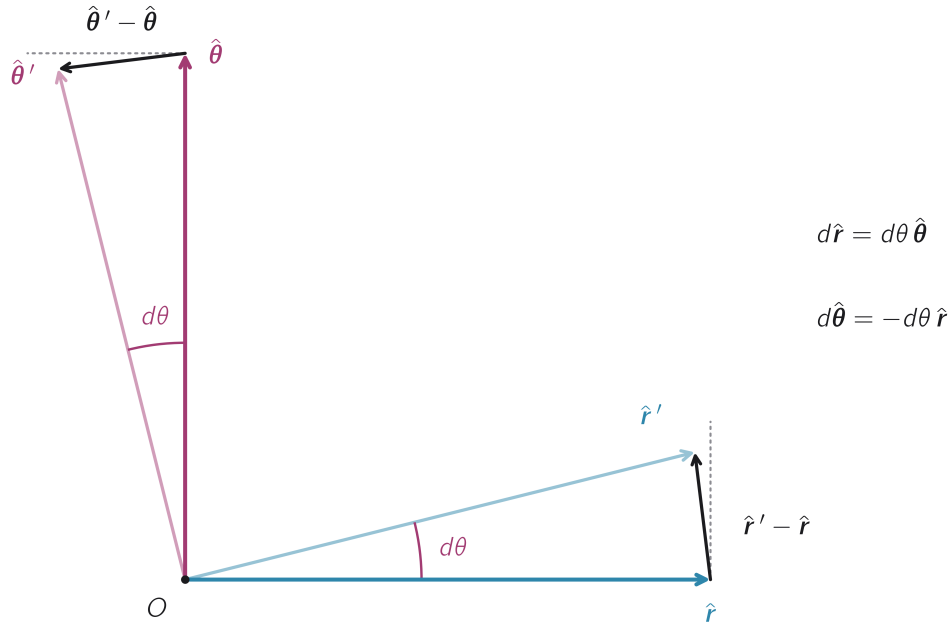


Figure 8: The unit vectors $\hat{r}$ and $\hat{\theta}$ start at a common origin, with copies $\hat{r}'$ and $\hat{\theta}'$ rotated through a small angle $d\theta$ (exaggerated). The finite arrows joining the tips show the limiting directions: as $d\theta \rightarrow 0$, $d\hat{r} = d\theta \hat{\theta}$ and $d\hat{\theta} = -d\theta \hat{r}$.

At what rate does $\hat{\theta}$ change? At the same rate as $\hat{r}$. The two vectors rotate jointly as a perpendicular pair; if one turned faster than the other, they would not stay perpendicular. Both therefore turn through the same angle, and their derivatives have the same magnitude, $|\dot{\theta}|$, with directions fixed by the sense of the turn:

$$\boxed{\frac{d\hat{r}}{dt} = \dot{\theta} \hat{\theta}, \quad \frac{d\hat{\theta}}{dt} = -\dot{\theta} \hat{r}.}$$

These identities hold for any smooth $\theta(t)$ in a fixed plane, not only for constant Ω .

For our circular motion, $\dot{\theta} = \Omega$. Substituting into the acceleration,

$$\mathbf{a}(t) = R\Omega \frac{d\hat{\theta}}{dt} = R\Omega(-\Omega \hat{r}), \quad \boxed{\mathbf{a}(t) = -R\Omega^2 \hat{r}(t).}$$

The acceleration points inward, toward the center, even though the speed is constant: it changes the direction of the velocity, not its length. Its radial component $-R\Omega^2$ and its magnitude $R\Omega^2$ are constant; the vector itself rotates with $\hat{r}$. Since $v = R\Omega$, the magnitude can also be written $R\Omega^2 = v^2/R$.

Comparing magnitudes, for $R, \Omega > 0$,

$$\frac{|\mathbf{a}|}{|\mathbf{v}|} = \frac{R\Omega^2}{R\Omega} = \Omega,$$

so this ratio does not depend on the radius. The minus sign in the vector acceleration gives the inward direction; it does not make the magnitude negative.

✓ Lengths and right angle are preserved

Differentiating with the pair of basis derivatives,

$$\begin{aligned}\frac{d}{dt}(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}) &= 2\dot{\theta} \hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = 0, \\ \frac{d}{dt}(\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}}) &= -2\dot{\theta} \hat{\boldsymbol{\theta}} \cdot \hat{\mathbf{r}} = 0, \\ \frac{d}{dt}(\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}}) &= \dot{\theta}(\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} - \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}) = 0.\end{aligned}$$

Both vectors stay unit length, and they stay perpendicular. The last line needs the minus sign in $d\hat{\boldsymbol{\theta}}/dt$.

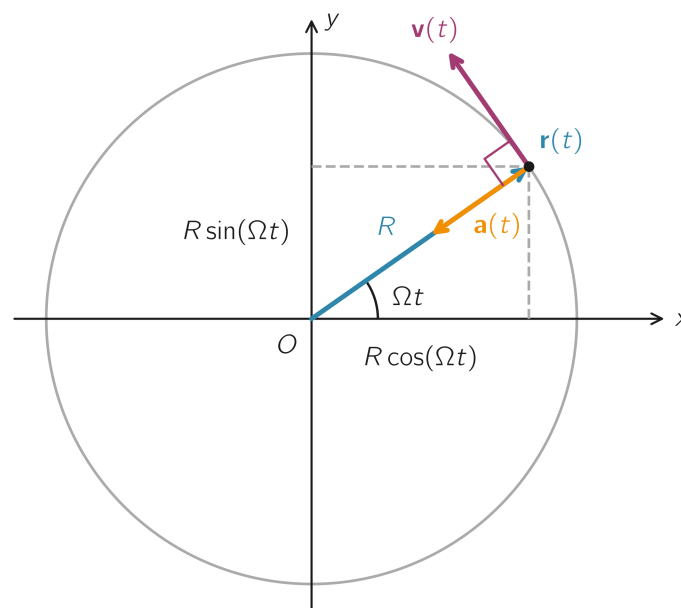


Figure 9: Position, velocity, and acceleration in uniform circular motion at one instant. The position $\mathbf{r}(t)$ (blue) has length R and angle Ωt ; its Cartesian projections are $R \cos(\Omega t)$ and $R \sin(\Omega t)$. The velocity $\mathbf{v}(t)$ (magenta) is perpendicular to the position, along $\hat{\boldsymbol{\theta}}$. The acceleration $\mathbf{a}(t)$ (orange) points from the particle toward the center, along $-\hat{\mathbf{r}}$. The three arrows carry different units and are drawn with separate scales; compare their directions, not their lengths.

⊙ Demonstration

Open demonstration ↗ Use the polar-basis demonstration to follow the rotating unit vectors. At fixed radius, doubling the angular rate quadruples the acceleration readout. The displayed clock is not a measure of the physical period.

4.1 Release from a circular path

✍ Release from a circular path

A ball on a string moves counterclockwise around a circle of radius R at constant angular rate Ω on a smooth horizontal table. At some instant the string breaks. Sketch the path afterwards: does the ball keep curving, move radially outward, or move along a straight line? In which direction?

Choose the coordinates to make the description simple. Put the release at $t = 0$, with the ball on the positive x axis, so that $\theta(t) = \Omega t$ before the release.

Before the release, $t < 0$. The ball is in uniform circular motion:

$$\begin{aligned}\mathbf{r}(t) &= R[\cos(\Omega t) \hat{\mathbf{x}} + \sin(\Omega t) \hat{\mathbf{y}}] = R\hat{\mathbf{r}}(t), \\ \mathbf{v}(t) &= R\Omega[-\sin(\Omega t) \hat{\mathbf{x}} + \cos(\Omega t) \hat{\mathbf{y}}] = R\Omega\hat{\boldsymbol{\theta}}(t), \\ \mathbf{a}(t) &= -R\Omega^2\hat{\mathbf{r}}(t).\end{aligned}$$

At the release, $t = 0$. Here $\hat{\mathbf{r}}(0) = \hat{\mathbf{x}}$ and $\hat{\boldsymbol{\theta}}(0) = \hat{\mathbf{y}}$, so

$$\mathbf{r}_0 = R\hat{\mathbf{x}}, \quad \mathbf{v}_0 = R\Omega\hat{\mathbf{y}}.$$

At this point of the circle the angular direction coincides with the y direction.

After the release, $t > 0$. Nothing holds the ball on the circle any more. We describe what follows by prescribing zero acceleration,

$$\ddot{\mathbf{r}} = \mathbf{0} \quad (t > 0),$$

the constant-velocity model of Week 2. Read the derivatives backwards. A velocity whose derivative is zero is constant, and it starts from $\mathbf{v}_0$:

$$\mathbf{v}(t) = \mathbf{v}_0 = R\Omega\hat{\mathbf{y}}.$$

A position whose derivative is the constant $R\Omega\hat{\mathbf{y}}$ changes linearly in time, starting from $\mathbf{r}_0$:

$$\boxed{\mathbf{r}(t) = R\hat{\mathbf{x}} + R\Omega t\hat{\mathbf{y}}, \quad t \geq 0.}$$

Differentiating checks both statements: $\dot{\mathbf{r}} = R\Omega\hat{\mathbf{y}}$ and $\ddot{\mathbf{r}} = \mathbf{0}$, and at $t = 0$ we recover $\mathbf{r}_0$ and $\mathbf{v}_0$. The ball moves along the vertical line $x = R$, the tangent to the circle at the release point, at the speed $R\Omega$ it already had.

Compare the two sides of $t = 0$:

	just before release	just after release
position	$R \hat{x}$	$R \hat{x}$
velocity	$R\Omega \hat{y}$	$R\Omega \hat{y}$
acceleration	$-R\Omega^2 \hat{x}$	$\mathbf{0}$

Position and velocity are continuous; only the acceleration jumps. The release does not kick the ball: its velocity is the same immediately before and after, so it leaves along the tangent with the speed it already had. What disappears is the inward acceleration, and with it the turning of the velocity.

⊙ Demonstration

Open demonstration ↗ Compare a predicted release path with the cut-the-string demonstration. Use it to examine the path geometry; its clock does not give the physical elapsed time.

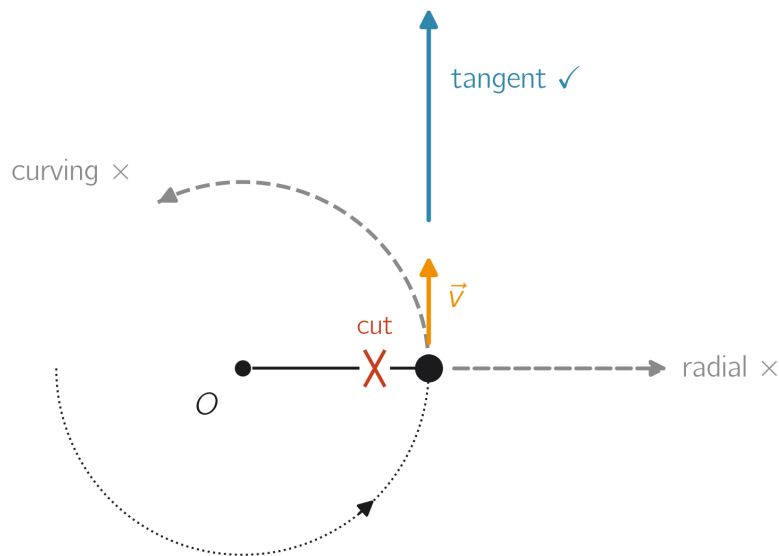


Figure 10: The ball sits at the rightmost point of its counterclockwise circle, $\mathbf{r}_0 = R \hat{x}$, when the string is cut. Its velocity there points straight up, $\mathbf{v}_0 = R\Omega \hat{y}$, and the ball carries on along that tangent. The radial path and the curving path are the two common wrong answers.

The released path is a straight line that does not pass through the origin. Along it, both the distance and the direction from the origin keep changing: r grows from R , and θ increases from 0 toward $\pi/2$. Circular motion held r fixed. To describe motion like this in the polar basis, we need both $r(t)$ and $\theta(t)$ to vary.

5 Velocity in Polar Coordinates

Now let both $r(t)$ and $\theta(t)$ change arbitrarily. Start from $\mathbf{r} = r \hat{\mathbf{r}}$ and use $\dot{\mathbf{r}} = \dot{\theta} \hat{\boldsymbol{\theta}}$:

$$\mathbf{v} = \frac{d}{dt}(r \hat{\mathbf{r}}) = \dot{r} \hat{\mathbf{r}} + r \dot{\hat{\mathbf{r}}} = \dot{r} \hat{\mathbf{r}} + r \dot{\theta} \hat{\boldsymbol{\theta}}.$$

$\mathbf{v} = \underbrace{\dot{r} \hat{\mathbf{r}}}_{\text{radial}} + \underbrace{r \dot{\theta} \hat{\boldsymbol{\theta}}}_{\text{tangential}}$
--

In Section 2 we split a small displacement into $d\mathbf{r} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}}$. Dividing by dt turns these lengths into rates and gives the velocity decomposition.

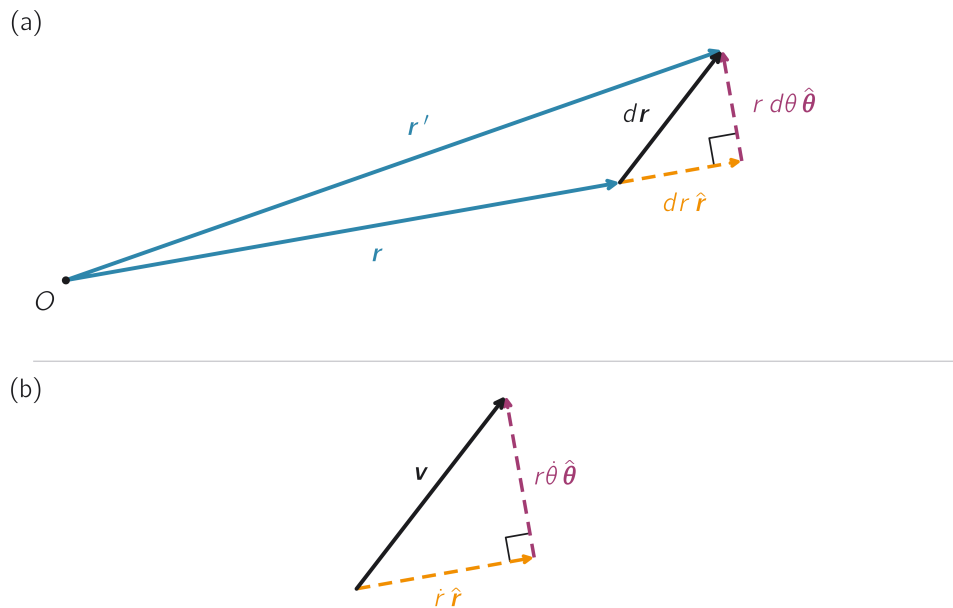


Figure 11: (a) To first order, the small displacement $d\mathbf{r}$ splits into a radial part $dr \hat{\mathbf{r}}$ and a perpendicular part $r d\theta \hat{\boldsymbol{\theta}}$, drawn on the same length scale with the infinitesimals exaggerated. (b) Dividing by dt gives $\mathbf{v} = \dot{r} \hat{\mathbf{r}} + r \dot{\theta} \hat{\boldsymbol{\theta}}$, drawn on its own velocity scale.

The components follow from dot products, as for any vector:

$$v_r = \mathbf{v} \cdot \hat{\mathbf{r}} = \dot{r}, \quad v_\theta = \mathbf{v} \cdot \hat{\boldsymbol{\theta}} = r \dot{\theta}.$$

- **Radial velocity** $v_r = \dot{r}$. It measures how fast the particle's distance from the origin changes.
- **Tangential velocity** $v_\theta = r \dot{\theta}$. It measures how fast the particle swings around the origin. The factor of r means that the same angular rate $\dot{\theta}$ gives a larger tangential speed at larger radius. A kid at the edge of a merry-go-round moves faster than one near the axle.

⚠ Tangent to which curve?

“Tangential” refers to $\hat{\theta}$, which is tangent to the circle centered at the origin through the particle. The full velocity $\mathbf{v}$ is tangent to the particle’s actual trajectory. For nonzero velocity, $\mathbf{v}$ is parallel to $\hat{\theta}$ exactly when $\dot{r} = 0$, pointing along or opposite to $\hat{\theta}$ according to the sign of $\dot{\theta}$. On an outward spiral, $\mathbf{v}$ also has a component along $\hat{r}$, and the path crosses the circles about the origin.

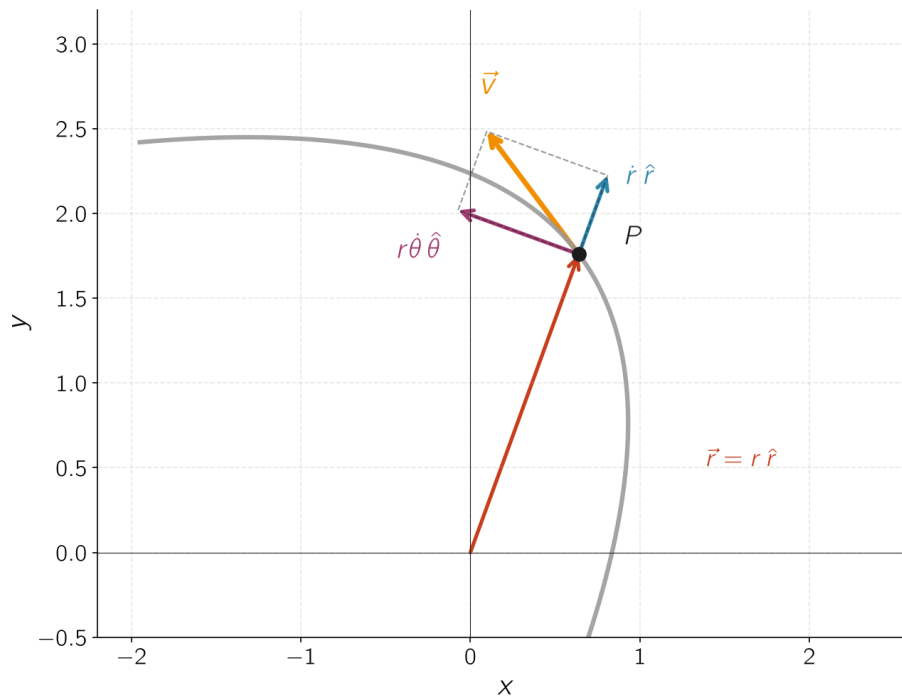


Figure 12: Velocity decomposed in polar coordinates. At the particle P the velocity $\mathbf{v}$ is tangent to the path. In the local polar basis it splits into a component $\dot{r}\hat{r}$ along $\hat{r}$ and a component $r\dot{\theta}\hat{\theta}$ perpendicular to it.

✓ Units — Two velocity components

$[\dot{r}] = \text{m/s}$. For $r\dot{\theta}$: $[r] = \text{m}$ and $[\dot{\theta}] = \text{rad/s} = \text{s}^{-1}$, since radians are dimensionless, so $[r\dot{\theta}] = \text{m/s}$. In both, the meter comes from r and the second from the time derivative. They are two different speeds: the rate at which the distance from the origin changes, and the distance times the rate at which the angle changes.

Special cases.

- Straight-line motion along a fixed radial line, away from $r = 0$, has $\dot{\theta} = 0$, so $\mathbf{v} = \dot{r}\hat{r}$ is purely radial.

- Circular motion at radius R has $\dot{r} = 0$, so $\mathbf{v} = R\dot{\theta}\hat{\boldsymbol{\theta}}$ is purely tangential and perpendicular to $\mathbf{r}$ at every instant. The condition $r = R$ removes the radial component but says nothing about $\dot{\theta}$: the velocity, not the position, carries how fast the particle goes around. For $\dot{\theta} = \Omega$ we recover $\mathbf{v} = R\Omega\hat{\boldsymbol{\theta}}$.

At the origin the polar direction is undefined, so a trajectory crossing it must be described separately on either side.

6 Acceleration in Polar Coordinates

In uniform circular motion only one term survived, $\mathbf{a} = -R\Omega^2\hat{\mathbf{r}}$. That is not the general result: it used $\dot{r} = 0$ and a constant $\dot{\theta}$. For the general acceleration, differentiate the general velocity. Both the components and the basis vectors now depend on time.

Σ Product rule

The product rule is unchanged when one factor is a vector:

$$\frac{d}{dt}[f(t)\mathbf{A}(t)] = \dot{f}(t)\mathbf{A}(t) + f(t)\dot{\mathbf{A}}(t).$$

The first term differentiates the component; the second differentiates the basis vector. For the angle dependence, use the chain rule from Section 2.1.

$\therefore$ Acceleration in a moving basis

Differentiate $\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$. Both the coefficients and the basis vectors depend on time, so begin by separating the two terms:

$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt} \\ &= \frac{d}{dt}(\dot{r}\hat{\mathbf{r}}) + \frac{d}{dt}(r\dot{\theta}\hat{\boldsymbol{\theta}}).\end{aligned}$$

The radial term requires one product rule:

$$\frac{d}{dt}(\dot{r}\hat{\mathbf{r}}) = \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\hat{\mathbf{r}}}.$$

The tangential term contains the product of scalars $r\dot{\theta}$ multiplying $\hat{\boldsymbol{\theta}}$. Differentiate the scalar first,

$$\frac{d}{dt}(r\dot{\theta}) = \dot{r}\dot{\theta} + r\ddot{\theta},$$

and then apply the product rule to the full term:

$$\begin{aligned}\frac{d}{dt}(r\dot{\theta}\hat{\boldsymbol{\theta}}) &= \frac{d}{dt}(r\dot{\theta})\hat{\boldsymbol{\theta}} + r\dot{\theta}\dot{\hat{\boldsymbol{\theta}}} \\ &= (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}} + r\dot{\theta}\dot{\hat{\boldsymbol{\theta}}}.\end{aligned}$$

Put the two pieces back together. Only now substitute $\dot{\hat{\mathbf{r}}} = \dot{\theta} \hat{\boldsymbol{\theta}}$ and $\dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta} \hat{\mathbf{r}}$:

$$\begin{aligned}\mathbf{a} &= \ddot{r} \hat{\mathbf{r}} + \dot{r} \dot{\hat{\mathbf{r}}} + (\dot{r}\dot{\theta} + r\ddot{\theta}) \hat{\boldsymbol{\theta}} + r\dot{\theta} \dot{\hat{\boldsymbol{\theta}}} \\ &= \ddot{r} \hat{\mathbf{r}} + \dot{r}(\dot{\theta} \hat{\boldsymbol{\theta}}) + (\dot{r}\dot{\theta} + r\ddot{\theta}) \hat{\boldsymbol{\theta}} + r\dot{\theta}(-\dot{\theta} \hat{\mathbf{r}}) \\ &= \ddot{r} \hat{\mathbf{r}} - r\dot{\theta}^2 \hat{\mathbf{r}} + \dot{r}\dot{\theta} \hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta} \hat{\boldsymbol{\theta}} + r\ddot{\theta} \hat{\boldsymbol{\theta}} \\ &= (\ddot{r} - r\dot{\theta}^2) \hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta}) \hat{\boldsymbol{\theta}}.\end{aligned}$$

The factor of 2 comes from the two identical terms $\dot{r}\dot{\theta} \hat{\boldsymbol{\theta}}$. One comes from differentiating the radial basis vector. The other comes from differentiating the scalar $r\dot{\theta}$.

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2) \hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \hat{\boldsymbol{\theta}}.$$

✓ Units — Four acceleration terms

Radians are dimensionless, so $[\dot{\theta}] = \text{s}^{-1}$ and $[\ddot{\theta}] = \text{s}^{-2}$. Then

$$[\ddot{r}] = \text{m/s}^2, \quad [r\dot{\theta}^2] = \text{m} \cdot \text{s}^{-2}, \quad [r\ddot{\theta}] = \text{m} \cdot \text{s}^{-2}, \quad [\dot{r}\dot{\theta}] = (\text{m/s})(\text{s}^{-1}) = \text{m/s}^2.$$

In every term the meter comes from r and the two inverse seconds from two time derivatives. All four have the dimensions of acceleration. The check cannot determine their signs or the factor of two.

The four contributions group into two components:

Term	Meaning
$\ddot{r} \hat{\mathbf{r}}$	Radial speed-up or slow-down
$-r\dot{\theta}^2 \hat{\mathbf{r}}$	Centripetal acceleration: inward, present whenever there is rotation
$r\ddot{\theta} \hat{\boldsymbol{\theta}}$	Tangential speed-up or slow-down from changing $\dot{\theta}$
$2\dot{r}\dot{\theta} \hat{\boldsymbol{\theta}}$	Coriolis term: present when there is <i>both</i> radial and angular motion

These are contributions to one acceleration, resolved in the moving basis. The term $2\dot{r}\dot{\theta}$ is called the **Coriolis term** (Kleppner and Kolenkow, §1.11); here it is not a new force.

✓ Uniform circular motion

For $r = R$ and $\dot{\theta} = \Omega$ constant, $\dot{r} = 0$, $\ddot{r} = 0$, and $\ddot{\theta} = 0$. Three of the four terms vanish:

$$\mathbf{a} = (0 - R\Omega^2) \hat{\mathbf{r}} + (R \cdot 0 + 2 \cdot 0 \cdot \Omega) \hat{\boldsymbol{\theta}} = -R\Omega^2 \hat{\mathbf{r}},$$

the inward acceleration found in Section 4.

7 Example: Outward Spiral

A bug walks straight outward along a radial line painted on a turntable. The table turns counterclockwise at the steady rate $\Omega > 0$. Relative to the table, the bug keeps a steady pace $u > 0$ along the line. At $t = 0$ the bug is a distance $r_0 > 0$ from the center, on the positive x axis, so $\theta_0 = 0$. The motion is prescribed: we are told how r and θ change, and we want the path, the velocity and the acceleration.

Read the words as equations of motion. “Moves outward with constant radial speed” and “turns at constant angular rate” say

$$\dot{r} = u, \quad \dot{\theta} = \Omega.$$

The first statement is about the distance r , a scalar, not about the vector $\mathbf{r}$. It says nothing about the direction of $\mathbf{r}$ and does not say that $\dot{\mathbf{r}}$ is constant. The turning comes from the second statement. These are first-order statements, so each needs one starting value. The setup supplies them:

$$r(0) = r_0, \quad \theta(0) = 0.$$

Draw the geometry. Sketch the x and y axes and mark r_0 on the positive x -axis. Starting there, draw a curve that winds counterclockwise and moves farther from the origin with every turn. Pick one point on the curve. At that point draw $\hat{\mathbf{r}}$ pointing directly away from the origin and $\hat{\boldsymbol{\theta}}$ at right angles to it, pointing counterclockwise. Then draw $\mathbf{v}$ between them, tilted outward from $\hat{\boldsymbol{\theta}}$. The spiral crosses the circle through that point instead of following it, so $\hat{\boldsymbol{\theta}}$ is not the tangent to the path.

Guess first

Both rates are constant. Is the acceleration zero? If not, which way does it point?

Find a function with the stated derivative. A function whose derivative is the constant u is ut plus a constant, and the initial values fix that constant:

$$r(t) = r_0 + ut, \quad \theta(t) = \Omega t.$$

Eliminate time using $t = \theta/\Omega$:

$$r = r_0 + \frac{u}{\Omega} \theta.$$

The radius grows linearly with angle. This is an Archimedean spiral, and successive turns are separated in radius by $2\pi u/\Omega$.

The velocity follows from $\mathbf{v} = \dot{r} \hat{\mathbf{r}} + r\dot{\theta} \hat{\boldsymbol{\theta}}$:

$$\mathbf{v} = u \hat{\mathbf{r}} + r\Omega \hat{\boldsymbol{\theta}}, \quad |\mathbf{v}| = \sqrt{u^2 + r^2\Omega^2}.$$

The bug's pace u is steady relative to the table, but its laboratory speed $\sqrt{u^2 + r^2\Omega^2}$ grows as it walks outward. The radial part u keeps the same size, but it points along $\hat{\mathbf{r}}$, which keeps turning with the table. The tangential part $r\Omega$ gets larger as r increases. So the laboratory

velocity changes in both direction and magnitude. A constant walking pace does not mean zero laboratory acceleration.

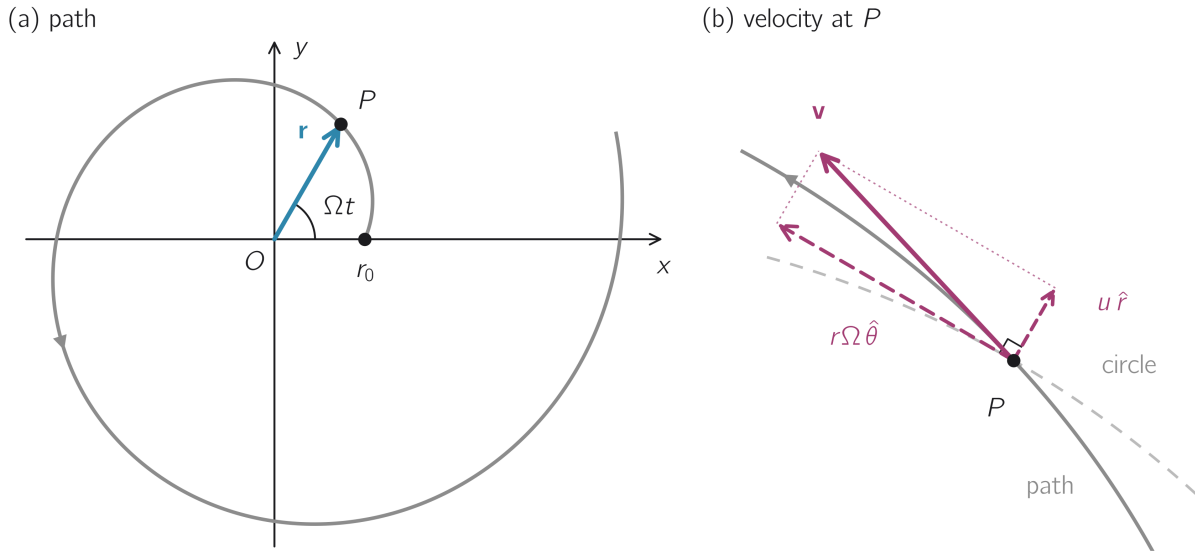


Figure 13: Outward spiral $r = r_0 + ut$, $\theta = \Omega t$. (a) The path starts at $(r_0, 0)$ and winds counterclockwise with growing radius; $\mathbf{r}$ is the position of one point P . (b) Close-up near P , all three arrows on one velocity scale: the components $u \hat{\mathbf{r}}$ and $r\Omega \hat{\boldsymbol{\theta}}$ add to $\mathbf{v}$. $\mathbf{v}$ is tangent to the path; $\hat{\boldsymbol{\theta}}$ is tangent to the dashed origin-centered circle through P , so $\mathbf{v}$ tilts outward from $\hat{\boldsymbol{\theta}}$.

The same velocity also comes out of Cartesian coordinates, where the unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are fixed in space. Only the components change. Each component is the projection of $\mathbf{r}$ onto a fixed axis:

$$\begin{aligned} x &= \mathbf{r} \cdot \hat{\mathbf{x}} = (r_0 + ut) \cos \theta(t), \\ y &= \mathbf{r} \cdot \hat{\mathbf{y}} = (r_0 + ut) \sin \theta(t), \quad \theta(t) = \Omega t. \end{aligned}$$

$$\mathbf{r}(t) = (r_0 + ut) [\cos(\Omega t) \hat{\mathbf{x}} + \sin(\Omega t) \hat{\mathbf{y}}].$$

Both components are products of two time-dependent factors, so we use the product rule. Each derivative has two terms:

$$\begin{aligned} \dot{x} &= \frac{d}{dt}(r_0 + ut) \cos(\Omega t) + (r_0 + ut) \frac{d}{dt} \cos(\Omega t) \\ &= u \cos(\Omega t) - (r_0 + ut) \Omega \sin(\Omega t), \\ \dot{y} &= \frac{d}{dt}(r_0 + ut) \sin(\Omega t) + (r_0 + ut) \frac{d}{dt} \sin(\Omega t) \\ &= u \sin(\Omega t) + (r_0 + ut) \Omega \cos(\Omega t). \end{aligned}$$

Because $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ do not change, the velocity is just these components put back on the fixed axes, with $r = r_0 + ut$:

$$\begin{aligned}\mathbf{v} &= \dot{x} \hat{\mathbf{x}} + \dot{y} \hat{\mathbf{y}} \\ &= [u \cos(\Omega t) - r\Omega \sin(\Omega t)] \hat{\mathbf{x}} \\ &\quad + [u \sin(\Omega t) + r\Omega \cos(\Omega t)] \hat{\mathbf{y}}.\end{aligned}$$

Now group the terms by what multiplies them:

$$\begin{aligned}\mathbf{v} &= u [\cos(\Omega t) \hat{\mathbf{x}} + \sin(\Omega t) \hat{\mathbf{y}}] \\ &\quad + r\Omega [-\sin(\Omega t) \hat{\mathbf{x}} + \cos(\Omega t) \hat{\mathbf{y}}] \\ &= u \hat{\mathbf{r}} + r\Omega \hat{\boldsymbol{\theta}}.\end{aligned}$$

The first bracket is $\hat{\mathbf{r}}$ and the second is $\hat{\boldsymbol{\theta}}$. This is the same velocity we found in polar coordinates. In polar form the two terms appeared directly: u comes from the radius growing, and $r\Omega$ comes from the turning of $\hat{\mathbf{r}}$. In Cartesian form the axes stay fixed, and that same physics is spread across the oscillating components $\dot{x}$ and $\dot{y}$. The answer is identical, but getting there takes more algebra.

For the acceleration, use $\ddot{r} = 0$ and $\ddot{\theta} = 0$ in the polar formula:

$$\begin{aligned}\mathbf{a} &= (\ddot{r} - r\dot{\theta}^2) \hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \hat{\boldsymbol{\theta}} \\ &= (0 - r\Omega^2) \hat{\mathbf{r}} + (0 + 2u\Omega) \hat{\boldsymbol{\theta}} \\ &= -r\Omega^2 \hat{\mathbf{r}} + 2u\Omega \hat{\boldsymbol{\theta}}.\end{aligned}$$

Constant rates do not give zero acceleration. The radial term points inward, as for circular motion at radius r with the same angular rate. The $2u\Omega$ term has two equal sources: differentiating $u \hat{\mathbf{r}}$ gives $u\Omega \hat{\boldsymbol{\theta}}$ because the radial direction rotates, and differentiating $r\Omega \hat{\boldsymbol{\theta}}$ gives $\dot{r}\Omega \hat{\boldsymbol{\theta}} = u\Omega \hat{\boldsymbol{\theta}}$ because the radius grows (the rotation of $\hat{\boldsymbol{\theta}}$ in that same product gives the $-r\Omega^2 \hat{\mathbf{r}}$ term).

Concept questions

1. Set $u \rightarrow 0$. What motion remains, and what is $\mathbf{a}$?
2. Set $\Omega \rightarrow 0$. What motion remains, and what is $\mathbf{a}$?
3. Compare the particle at radius r with the same particle later at radius $2r$. How do the two components of $\mathbf{a}$ change?

Answers. (1) The particle moves on a circle of fixed radius r_0 at constant angular rate, with $\mathbf{a} = -r_0\Omega^2 \hat{\mathbf{r}}$: uniform circular motion. (2) Then $\theta = 0$ and $r = r_0 + ut$, straight outward motion at constant speed u , with $\mathbf{a} = \mathbf{0}$. (3) The radial component doubles, from $-r\Omega^2$ to $-2r\Omega^2$. The angular component $2u\Omega$ does not depend on r and stays the same.

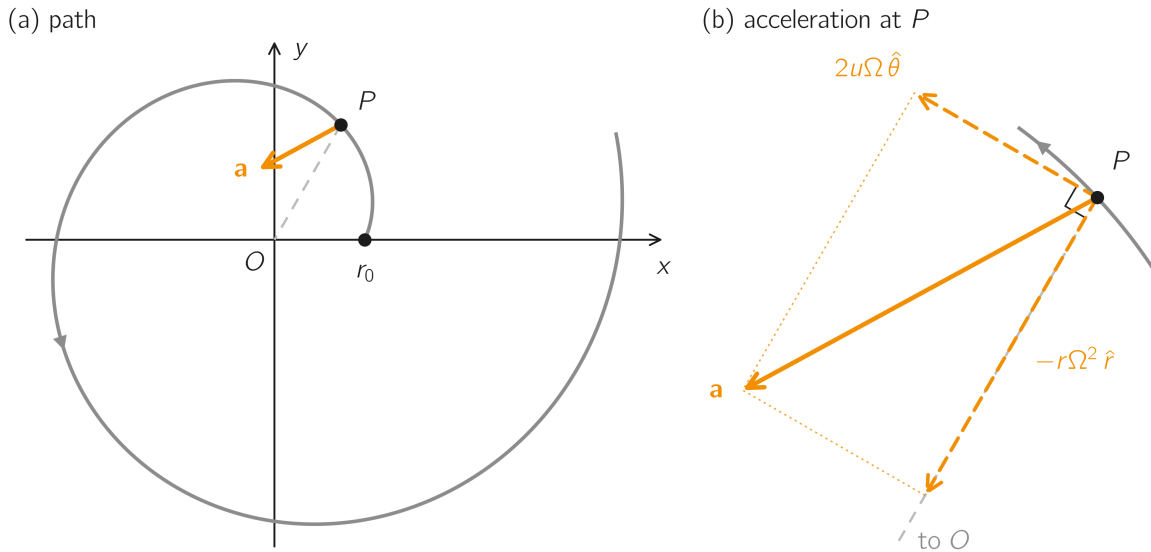


Figure 14: Acceleration on the outward spiral $r = r_0 + ut$, $\theta = \Omega t$, at the same point P ($\theta = 60^\circ$) as the velocity figure. (a) The net acceleration $\mathbf{a}$ at P ; the dashed line runs from P to the origin, and $\mathbf{a}$ does not lie along it. (b) Magnified view at P , all three arrows on one acceleration scale: the inward radial part $-r\Omega^2 \hat{\mathbf{r}}$ and the angular part $2u\Omega \hat{\boldsymbol{\theta}}$, drawn from P , add to $\mathbf{a}$. Because $2u\Omega > 0$, $\mathbf{a}$ tilts from the inward radial direction toward $+\hat{\boldsymbol{\theta}}$, so it does not point at the origin.

Starting at the center. Put the bug at the center at $t = 0$, so $r_0 = 0$. Then $r = ut$, the spiral is $r = (u/\Omega)\theta$, and

$$\mathbf{r}(t) = ut [\cos(\Omega t) \hat{\mathbf{x}} + \sin(\Omega t) \hat{\mathbf{y}}].$$

The polar components of the velocity are u and $u\Omega t$, so

$$|\mathbf{v}| = \sqrt{u^2 + u^2\Omega^2 t^2} = u\sqrt{1 + \Omega^2 t^2} = u\sqrt{1 + \theta^2}, \quad \mathbf{a} = -u\Omega^2 t \hat{\mathbf{r}} + 2u\Omega \hat{\boldsymbol{\theta}}.$$

The speed grows without bound even though $\dot{r}$ and $\dot{\theta}$ never change. Much of the time dependence of $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{a}$ is carried by the basis vectors $\hat{\mathbf{r}}(t)$ and $\hat{\boldsymbol{\theta}}(t)$, not by the coefficients.

Next week. In uniform circular motion the acceleration points back toward the center and grows in proportion to the distance from it: $\mathbf{a} = -R\Omega^2 \hat{\mathbf{r}} = -\Omega^2 \mathbf{r}$. Read as a rule, this says that at every instant the acceleration is fixed by the position. Is a circle the only path that obeys it?

Summary

Rotating basis. The unit vectors $\hat{\mathbf{r}}$ (radially outward) and $\hat{\boldsymbol{\theta}} = \hat{\mathbf{n}} \times \hat{\mathbf{r}}$ (toward increasing θ , tangent to the circle about the origin) are attached to the particle. They rotate together as the angle changes:

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta} \hat{\boldsymbol{\theta}}, \quad \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\dot{\theta} \hat{\mathbf{r}}.$$

Uniform circular motion ($r = R$, $\dot{\theta} = \Omega = \text{const}$). A fixed radius alone does not specify the motion; the angular rate is needed as well. With $\theta(0) = 0$,

$$\mathbf{r} = R \hat{\mathbf{r}}, \quad \mathbf{v} = R\Omega \hat{\boldsymbol{\theta}}, \quad \mathbf{a} = -R\Omega^2 \hat{\mathbf{r}} = -\frac{v^2}{R} \hat{\mathbf{r}}, \quad T = \frac{2\pi}{\Omega}.$$

Release from a circle. With $\mathbf{r}_0 = R \hat{\mathbf{x}}$ and $\mathbf{v}_0 = R\Omega \hat{\mathbf{y}}$ at release, zero acceleration afterwards gives $\mathbf{r}(t) = R \hat{\mathbf{x}} + R\Omega t \hat{\mathbf{y}}$. Position and velocity are continuous; the acceleration jumps from $-R\Omega^2 \hat{\mathbf{x}}$ to zero.

General velocity and acceleration.

$$\mathbf{v} = \dot{r} \hat{\mathbf{r}} + r\dot{\theta} \hat{\boldsymbol{\theta}}, \quad \mathbf{a} = (\ddot{r} - r\dot{\theta}^2) \hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \hat{\boldsymbol{\theta}}.$$

The components are dot products with the basis: $v_r = \dot{r}$, $v_\theta = r\dot{\theta}$, and $a_r = \ddot{r} - r\dot{\theta}^2$, which is not $\ddot{r}$ alone.

Outward spiral. With $\dot{r} = u$ and $\dot{\theta} = \Omega$ constant, $r = r_0 + ut$ and $\theta = \Omega t$:

$$\mathbf{v} = u \hat{\mathbf{r}} + r\Omega \hat{\boldsymbol{\theta}}, \quad \mathbf{a} = -r\Omega^2 \hat{\mathbf{r}} + 2u\Omega \hat{\boldsymbol{\theta}}.$$

Constant rates $\dot{r}$ and $\dot{\theta}$ do not give zero acceleration.