

Physics I: Mechanics and Relativity

Newton's Laws and Universal Gravitation

Week 04

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Readings

1. **Feynman Lectures I, Ch. 9, §§9-1 and 9-4.** Momentum, force, and the distinction between Newton's second law and a force law. §9-7 develops numerical orbital motion. [Feynman, Chapter 9 ↗](#)
2. **Feynman Lectures I, Ch. 7, §§7-2–7-4.** Inertia, inward force, Kepler's laws and the comparison of gravity at the Earth's surface with the Moon's acceleration. [Feynman, Chapter 7 ↗](#)
3. **Feynman Lectures I, Ch. 12, §12-1.** Why defining force as $m\mathbf{a}$ alone makes no prediction. [Feynman, Chapter 12 ↗](#)
4. **Kleppner & Kolenkow, §§2.1–2.6 and §3.3.** Newton's laws and gravitation.
5. **Cole, course notes.** Newton's laws.
6. **E. P. Wigner, *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*,** Communications on Pure and Applied Mathematics **13**, 1–14 (1960). [Wigner's essay ↗](#)

Overview

Last week we described motion: given a trajectory, we computed its velocity and acceleration, and uniform circular motion gave an acceleration pointing back toward the center. Written as a differential equation, $\ddot{\mathbf{r}} = -\Omega^2 \mathbf{r}$, that result also produces ellipses. The planets move on ellipses, which raises the question kinematics cannot answer: why do bodies move as they do? Following the route from Aristotle's push through Tycho Brahe's measurements and Kepler's laws to inertia, we find the Sun's inward, inverse-square pull, then Newton's gravitation and laws of motion. We end by building an orbit numerically, one short time step at a time.

1 The Equation of Uniform Circular Motion

Last week, uniform circular motion with radius R and constant angular rate Ω gave

$$\mathbf{r} = R \hat{\mathbf{r}}, \quad \mathbf{v} = R\Omega \hat{\boldsymbol{\theta}}, \quad \mathbf{a} = -R\Omega^2 \hat{\mathbf{r}}.$$

Each time derivative turns the vector through a right angle: $\mathbf{v}$ is perpendicular to $\mathbf{r}$, and $\mathbf{a}$ to $\mathbf{v}$, so the acceleration points opposite to the position. Since $\mathbf{r} = R\hat{\mathbf{r}}$, the acceleration is $-\Omega^2$ times the position itself. The units agree: the factor between an acceleration in m/s^2 and a position in meters carries s^{-2} , two powers of Ω . With the dot notation $\dot{\mathbf{r}} \equiv d\mathbf{r}/dt$ and $\ddot{\mathbf{r}} \equiv d^2\mathbf{r}/dt^2$,

$$\ddot{\mathbf{r}} = -\Omega^2 \mathbf{r}.$$

Differential equation for uniform circular motion. This one statement carries both the inward direction and the magnitude $R\Omega^2$. It mentions neither R nor the angle Ωt : it fixes the acceleration at each instant from the position at that instant.

1.1 Does inward acceleration mean a circle?

Guess first

Does the rule $\ddot{\mathbf{r}} = -\Omega^2 \mathbf{r}$, with $\Omega > 0$ fixed, force the path to be a circle?

Write the vector equation in the fixed Cartesian basis. Since $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ do not change, $\ddot{\mathbf{r}} = \ddot{x}\hat{\mathbf{x}} + \ddot{y}\hat{\mathbf{y}}$, and matching components gives two separate equations:

$$\ddot{x} = -\Omega^2 x, \quad \ddot{y} = -\Omega^2 y.$$

Each component obeys $\ddot{q} = -\Omega^2 q$, where q is x or y . Which functions return $-\Omega^2$ times themselves after two derivatives? Cosine and sine:

$$\begin{aligned} \frac{d}{dt} \cos \Omega t &= -\Omega \sin \Omega t, & \frac{d^2}{dt^2} \cos \Omega t &= -\Omega^2 \cos \Omega t, \\ \frac{d}{dt} \sin \Omega t &= \Omega \cos \Omega t, & \frac{d^2}{dt^2} \sin \Omega t &= -\Omega^2 \sin \Omega t. \end{aligned}$$

The equation is linear: second derivatives add and constants factor out, so any combination of the two also works. Each Cartesian component has its own two constants:

$$\begin{aligned} x(t) &= a \cos \Omega t + c \sin \Omega t, \\ y(t) &= b \cos \Omega t + d \sin \Omega t. \end{aligned}$$

A phase adds nothing new: by the angle-addition formula of Week 3, $\cos(\Omega t + \phi) = \cos \phi \cos \Omega t - \sin \phi \sin \Omega t$ is again such a combination.

Σ Exponential form

You could instead have guessed exponentials:

$$x(t) = \alpha e^{i\Omega t} + \beta e^{-i\Omega t}.$$

Two derivatives of $e^{\pm i\Omega t}$ bring down $(\pm i\Omega)^2 = -\Omega^2$, so each exponential solves $\ddot{q} = -\Omega^2 q$ on its own. To connect with cosines and sines, expand with Euler's formula from Week 3, $e^{\pm i\Omega t} = \cos \Omega t \pm i \sin \Omega t$, and collect terms:

$$x(t) = \alpha(\cos \Omega t + i \sin \Omega t) + \beta(\cos \Omega t - i \sin \Omega t) = (\alpha + \beta) \cos \Omega t + i(\alpha - \beta) \sin \Omega t.$$

Comparing with $x(t) = a \cos \Omega t + c \sin \Omega t$,

$$a = \alpha + \beta, \quad c = i(\alpha - \beta) \quad \Longleftrightarrow \quad \alpha = \frac{1}{2}(a - ic), \quad \beta = \frac{1}{2}(a + ic).$$

The coefficients α and β are complex in general. A real coordinate needs real a and c , which gives $\beta = \alpha^*$: the pair carries only two real constants, the real and imaginary parts of α . The same calculation for y uses b and d in place of a and c .

What fixes the four constants? The integration constants are fixed by information we know about the motion, such as the position and velocity at a chosen time. These are called **initial conditions**. The chosen time need not be the first moment of the motion; we can call it $t = 0$. Evaluate the solution and its derivative there:

$$\begin{aligned} x(0) &= a, & \dot{x}(0) &= \Omega c, \\ y(0) &= b, & \dot{y}(0) &= \Omega d. \end{aligned}$$

The initial position fixes a and b ; the initial velocity fixes c and d . Since any initial position and velocity can be matched, this family contains every solution in the plane, and the equation together with the initial data determines the motion.

Start the particle at $\mathbf{r}(0) = A \hat{\mathbf{x}}$ with $\mathbf{v}(0) = B\Omega \hat{\mathbf{y}}$, where $A > B > 0$. These initial values give

$$a = A, \quad b = 0, \quad c = 0, \quad d = B.$$

Therefore

$$\begin{aligned} \mathbf{r}(t) &= A \cos \Omega t \hat{\mathbf{x}} + B \sin \Omega t \hat{\mathbf{y}}, \\ x(t) &= A \cos \Omega t, & y(t) &= B \sin \Omega t. \end{aligned}$$

To eliminate time, divide by A and B , square, and add, using $\cos^2 \Omega t + \sin^2 \Omega t = 1$:

$$\begin{aligned} \frac{x}{A} &= \cos \Omega t, & \frac{y}{B} &= \sin \Omega t, \\ \frac{x^2}{A^2} &= \cos^2 \Omega t, & \frac{y^2}{B^2} &= \sin^2 \Omega t, \\ \frac{x^2}{A^2} + \frac{y^2}{B^2} &= \cos^2 \Omega t + \sin^2 \Omega t = 1. \end{aligned}$$

The path is an ellipse centered on the origin, with semi-axes A along $\hat{x}$ and B along $\hat{y}$. It is a circle only when $A = B$.

✓ **Same equation, different trajectories**

The circle and the ellipse obey the same equation, $\ddot{\mathbf{r}} = -\Omega^2 \mathbf{r}$, with the same Ω ; in both, the acceleration points opposite the position vector with magnitude $\Omega^2 r$. The initial conditions select the trajectory.

Sketch it. Draw the ellipse with semi-major axis A along x and semi-minor axis B along y . At $(A, 0)$ draw the velocity pointing up; at $(0, B)$ draw it pointing left. At both points draw the acceleration pointing toward the origin. With $B > 0$ the particle goes counterclockwise.

The velocity is

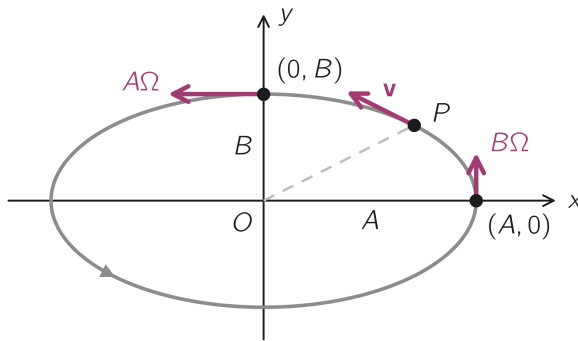
$$\mathbf{v}(t) = -A\Omega \sin \Omega t \hat{x} + B\Omega \cos \Omega t \hat{y},$$

so the speed satisfies

$$v^2 = \Omega^2 (A^2 \sin^2 \Omega t + B^2 \cos^2 \Omega t).$$

At the farthest point $(A, 0)$ the speed is $B\Omega$ and the acceleration has magnitude $A\Omega^2$. At the nearest point $(0, B)$ the speed is $A\Omega$ and the acceleration has magnitude $B\Omega^2$. The particle is fastest closest to the center, where the acceleration is smallest. Between these points the acceleration has a component along the velocity: it speeds the particle up on the way in toward the ends of the minor axis and slows it down on the way out toward the ends of the major axis. The *law* is the same everywhere; the value of the acceleration changes from point to point.

(a) velocity



(b) acceleration

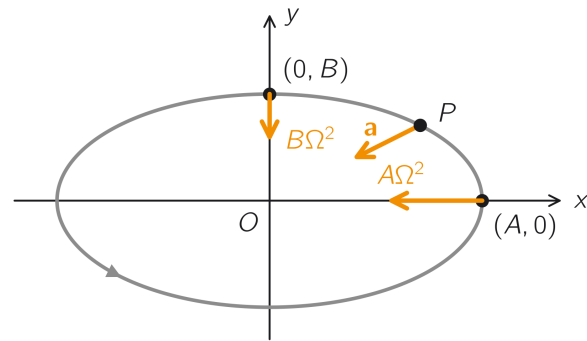


Figure 1: Ellipse $x = A \cos \Omega t$, $y = B \sin \Omega t$ for $A = 2B$, traversed counterclockwise. (a) Velocity: $B\Omega$ upward at $(A, 0)$, $A\Omega$ leftward at $(0, B)$. At P ($\Omega t = \pi/4$), $\mathbf{v}$ is tangent to the ellipse but not perpendicular to the dashed radius OP . (b) Acceleration: toward O everywhere, $A\Omega^2$ at $(A, 0)$ and $B\Omega^2$ at $(0, B)$; at P , $\mathbf{a} \cdot \mathbf{v} > 0$, so the speed increases toward $(0, B)$. Within each panel all arrows share one scale, so the endpoint arrows are in the true ratio A/B ; the two panels use independent scales and cannot be compared in length.

✍ **Guess first**

1. The phase Ωt grows at a constant rate. Does the polar angle θ ? If not, is $\dot{\theta}$ larger at $(A, 0)$ or at $(0, B)$?
2. At the ends of the axes the velocity is perpendicular to $\mathbf{r}$, so its tangential component v_θ is the whole speed: $B\Omega$ at $r = A$, and $A\Omega$ at $r = B$. Guess v_θ at a general point, built from A , B , Ω and r , in m/s.

Polar coordinates. Week 3 gives the velocity once we know $r(t)$ and $\theta(t)$:

$$\mathbf{v} = \dot{r} \hat{\mathbf{r}} + r \dot{\theta} \hat{\boldsymbol{\theta}}.$$

Both follow from $x = r \cos \theta = A \cos \Omega t$ and $y = r \sin \theta = B \sin \Omega t$. Squaring and adding gives the distance; dividing gives the angle:

$$r^2 = A^2 \cos^2 \Omega t + B^2 \sin^2 \Omega t, \quad \tan \theta = \frac{B}{A} \tan \Omega t.$$

Now differentiate each. For the distance, $2r\dot{r} = -2A^2\Omega \cos \Omega t \sin \Omega t + 2B^2\Omega \sin \Omega t \cos \Omega t$, so

$$\dot{r} = -\frac{(A^2 - B^2) \Omega \sin \Omega t \cos \Omega t}{r}.$$

For the angle, $d(\tan \theta)/dt = \dot{\theta}/\cos^2 \theta$, so $\dot{\theta}/\cos^2 \theta = (B/A)\Omega/\cos^2 \Omega t$. With $\cos \theta = x/r = (A/r) \cos \Omega t$,

$$\dot{\theta} = \frac{B}{A} \Omega \frac{\cos^2 \theta}{\cos^2 \Omega t} = \frac{B}{A} \Omega \frac{A^2}{r^2} = \frac{AB\Omega}{r^2}.$$

The velocity in the polar basis is therefore

$$\mathbf{v} = -\frac{(A^2 - B^2) \Omega \sin \Omega t \cos \Omega t}{r} \hat{\mathbf{r}} + \frac{AB\Omega}{r} \hat{\boldsymbol{\theta}}.$$

✓ **Polar velocity on the ellipse**

Both components are Ω times a length squared over a length: m/s. A circle, $A = B = R$, gives $\mathbf{v} = R\Omega \hat{\boldsymbol{\theta}}$, the Week 3 result. At $(A, 0)$, $\dot{r} = 0$ and $r\dot{\theta} = B\Omega$; at $(0, B)$, $\dot{r} = 0$ and $r\dot{\theta} = A\Omega$, the speeds found above. Squaring and adding the two components reproduces $v^2 = \Omega^2 (A^2 \sin^2 \Omega t + B^2 \cos^2 \Omega t)$.

For $A > B$ and $0 < \Omega t < \pi/2$, $\dot{r} < 0$: the particle moves inward on its way from $(A, 0)$ to $(0, B)$. The radial component vanishes only at the ends of the axes, where r is largest or smallest. Elsewhere the velocity, tangent to the ellipse, also has a radial part; $\hat{\boldsymbol{\theta}}$ is tangent to the circle of radius r about the origin, which is generally not the ellipse.

The angular rate $\dot{\theta} = AB\Omega/r^2$ is largest at the ends of the minor axis ($r = B$) and smallest at the ends of the major axis ($r = A$). The phase Ωt advances at the constant rate Ω ; the polar angle θ does not, and the two coincide throughout the motion only for a circle, $A = B$.

Guess first

The acceleration $-\Omega^2 \mathbf{r}$ points along $\mathbf{r}$. Which polar component of $\ddot{\mathbf{r}} = -\Omega^2 \mathbf{r}$ is zero?

With the Week 3 acceleration, $\mathbf{a} = (\ddot{r} - r\dot{\theta}^2) \hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \hat{\boldsymbol{\theta}}$, the law becomes

$$\ddot{r} - r\dot{\theta}^2 = -\Omega^2 r, \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.$$

On the circle, $r = R$, $\dot{r} = 0$ and $\dot{\theta} = \Omega$ give $\mathbf{a} = -R\Omega^2 \hat{\mathbf{r}}$, and both equations hold. In Cartesian form the law splits into independent equations for x and y ; in polar form r and θ appear in both, which is why we solved it in Cartesian coordinates.

1.2 What stays constant?

On a circle the ratio of acceleration to speed is the same at every point, $|\mathbf{a}|/|\mathbf{v}| = \Omega$, whenever we measure it. On the ellipse this ratio changes along the path:

$$\text{at } (A, 0) : \frac{|\mathbf{a}|}{|\mathbf{v}|} = \frac{A\Omega^2}{B\Omega} = \Omega \frac{A}{B}, \quad \text{at } (0, B) : \frac{|\mathbf{a}|}{|\mathbf{v}|} = \frac{B\Omega^2}{A\Omega} = \Omega \frac{B}{A}.$$

That constant belonged to the circle, not to the equation. Along the ellipse r , v and $|\mathbf{a}|$ all change. The ratio $|\mathbf{a}|/r = \Omega^2$ stays fixed, but that is no discovery: we put Ω into the rule.

Bodies that move on ellipses exist: the planets. Kepler found that each planet sweeps equal areas in equal times, but a planet's ellipse has the Sun at a focus, not at its center ([Section 2.3](#)). Whether the planets obey $\ddot{\mathbf{r}} = -\Omega^2 \mathbf{r}$, or some other rule, is a question mathematics cannot settle.

2 What Determines the Motion?

So far we have been doing **kinematics**. The outward spiral of Week 3 began from a prescription, $\dot{r} = u$ and $\dot{\theta} = \Omega$; the ellipse of [Section 1](#) began from a prescribed acceleration, $\ddot{\mathbf{r}} = -\Omega^2 \mathbf{r}$. Either way the rule came first, and we worked out the motion it implies. Nothing in kinematics says why a rule holds. **Dynamics** asks which physical interactions determine an object's acceleration, and how. Many candidate equations are mathematically consistent, and solving one tells us what would happen if nature obeyed it, not whether nature does. Mathematics is a language that allows us to reason consistently and work out the consequences of a rule. Physics is an experimental science: which rule holds is decided by observing how things actually move.

Mathematics and physics

In 1960 the physicist Eugene Wigner published an essay, *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*. One of his examples is Newton's law of gravitation: proposed on the basis of rough observations of falling bodies and planets, it proved accurate far beyond the precision of the data that suggested it. Once a mathematical law describes a set of observations, its further consequences become predictions, tested by new observations.

2.1 Two explanations of one observation

Push a piece of chalk along the table and let go. It slides and stops. For about two thousand years the standard account was Aristotle's: a body keeps moving only while something keeps moving it, and when the push ends, the motion ends. In this picture the sky needs a mover too: Aristotle placed the Earth at the center of nested celestial spheres that carried the Sun, the Moon and the planets around it.

The same observation has a second explanation: the table stops the chalk. Friction between the chalk and the table acts on the chalk, and that interaction, not the absence of a push, brings it to rest. Both explanations predict that the chalk stops, and logic alone cannot choose between them. An experiment can. Make the surface smoother and the chalk slides farther; a puck on ice or a glider on an air track coasts a long way. If motion simply died out when the push ended, smoothing the surface would make no difference.

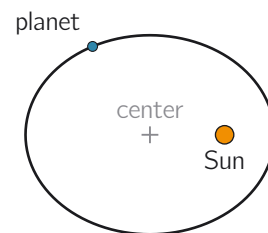
2.2 From observations to laws

Choosing between such explanations took precise data. In the late 1500s Tycho Brahe measured the positions of the planets night after night for decades, without a telescope and more accurately than anyone before him. A table of positions records where the planets were, not why. Johannes Kepler, who worked with Tycho and inherited his records, spent years asking what the trajectories are and what they all have in common; he published his first two laws in 1609 and the third in 1619. In the same decades Galileo Galilei studied falling bodies and balls rolling on ramps (Section 3). Isaac Newton combined these threads in his *Principia* of 1687: the same equations describe falling apples and orbiting planets, and they predicted motions not yet observed. They held for more than two centuries, until experiments at speeds close to that of light, and the slow turning of Mercury's orbit, required Einstein's relativity, which we will get to!

2.3 Kepler's laws

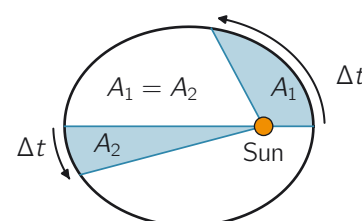
Kepler drew his laws from planetary observations, especially Tycho Brahe's measurements of Mars. With the Sun fixed and interactions between planets neglected, they read:

I. Law of Orbits. Each planet moves in an ellipse with the Sun at one focus.



II. Law of Areas. The area A swept by the line joining the Sun to the planet grows at a constant rate (here A is an area, not the semi-axis of Section 1):

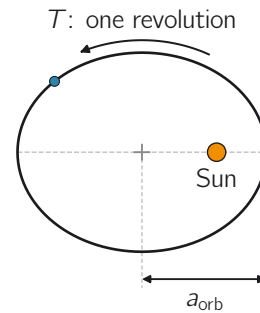
$$\dot{A} \equiv \frac{dA}{dt} = \text{constant}.$$



III. Law of Periods. Planets orbiting the same central body obey

$$\frac{T^2}{a_{\text{orb}}^3} = \text{constant},$$

where T is the orbital period and a_{orb} is the semi-major axis, half the longest diameter of the ellipse. The subscript distinguishes it from acceleration.



☉ Demonstration

Open the demonstration ➤ Explore an elliptical Earth orbit. Perigee and apogee are its closest and farthest points from Earth's center; changing the eccentricity changes how elongated the ellipse is.

Kepler's second law follows one planet along one orbit: $\dot{A}$ stays constant in time. His third law compares different orbits about the same central mass: T and a_{orb} vary, while T^2/a_{orb}^3 is shared.

Kepler's laws describe planetary motion; they do not say what produces it. Before asking what acts on a planet, we ask what a body does when nothing acts on it.

3 Principle of Inertia

Galileo reasoned from a ball rolling down one smooth ramp and up a second. The ball rises to nearly its starting height, and the shallower the second ramp, the farther it rolls to regain that height. Make the second ramp level and nothing remains to stop it. In his *Letters on Sunspots* (1613) Galileo stated that a body set moving on a smooth surface concentric with the Earth keeps moving without a further push; in that formulation the persistent motion follows the surface, around the Earth. Descartes, in his *Principles of Philosophy* (1644), stated explicitly that a body left to itself continues in a straight line.

☉ Demonstration

Open the demonstration ➤ In the cut-string demonstration of Week 3 the ball moves at constant velocity along the tangent once the inward force is gone.

Nothing pushes the ball forward after the string is cut; it keeps the velocity it had at that instant. Motion persists on its own. What needs an explanation is a *change* in the velocity. Section 6 states this principle formally, after we use it to find the pull of the Sun.

4 Inward Force and the Sun

4.1 Force

The velocity is changing. Who is to blame? By inertia, a body does not change its velocity on its own, and a change of direction needs an explanation just as much as a change of speed. We look for an interaction with something else (a string, a table, the Sun) that pushes or pulls. We call that interaction a *force*. For a planet, the question becomes: which body is pulling it, and in what direction?

Start with what we know. For uniform circular motion with constant radius R and constant angular rate Ω ,

$$\mathbf{a} = -R\Omega^2 \hat{\mathbf{r}}.$$

Guess first

An agent must keep a puck moving uniformly around this circle. Should it push along the tangent, in the direction of motion, or toward the center?

Connecting acceleration to a push takes a physical statement: experiments show that the net effect of the interactions points along the acceleration, and Newton's second law will make this quantitative. Here the acceleration points inward with no tangential component, so whatever keeps the puck on the circle must act toward the center, and nothing is needed along the motion. At release, with no net influence left, the puck leaves along the tangent and coasts in a straight line at constant speed. The ellipse adds a distinction: a pull toward the origin is generally not perpendicular to the path, so even with zero component along $\hat{\boldsymbol{\theta}}$ it changes the speed as well as the direction.

This contradicts Aristotle's intuition (Section 2.1) that motion needs a push to keep going. In Kepler's time one could picture invisible agents pushing the planets along their orbits. On a circular orbit a forward push would change the speed; the net pull must point inward.

Force cannot simply be *defined* as $m\mathbf{a}$: that would relabel the acceleration and predict nothing. A law of motion has content only when the force can be determined independently, for example from the stretch of a spring or the distance to the Sun, and when the motion it predicts keeps agreeing with what we observe.

For the planets, the angels would have to push toward the Sun. Newton proposed that no separate agent is needed: **the Sun itself is the source of the inward pull**. In the *Principia* (1687), Book III, Proposition V, he writes, in a literal translation from his Latin: “Jupiter’s planets gravitate toward Jupiter, and the circumsolar planets toward the Sun” (**original Latin** ↗).

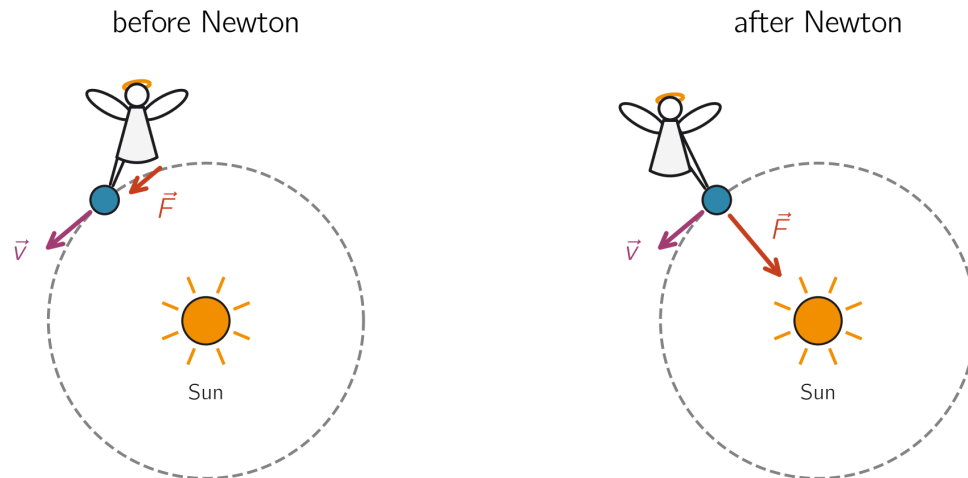


Figure 2: Feynman’s angel analogy contrasts a tangential push (left) with the inward force required for uniform circular motion (right). A tangential force changes the speed; an inward force changes the velocity’s direction. The angels are a cartoon, not a mechanism for gravity.

4.2 Areal rate and radial acceleration

How do we know the pull is radial? Newton showed that, in a fixed orbital plane, a radial pull and equal areas in equal times go together: each implies the other. That is the striking connection. Watching the areas tells us the direction of the acceleration.

In the *Principia* (1687), Book I, Proposition II, Newton writes that a body which “describes areas about that point proportional to the times is urged by a centripetal force directed toward that same point” (literal translation; [original Latin ↗](#)). Proposition I establishes the other direction. Let’s prove it to ourselves, keeping the Sun fixed and following the planet in its orbital plane.

∴ Areal rate and radial acceleration

Draw $\mathbf{r}$ from the Sun to the planet. It sweeps out an area $A(t)$. The area keeps growing, but Kepler tells us its rate $\dot{A} = dA/dt$, the *areal rate*, stays constant.

In a time dt , the planet moves by $d\mathbf{r} = \mathbf{v} dt$. The swept area is a thin triangle. The cross product gives the whole parallelogram, so we take half.

Divide by dt to get the rate:

$$dA = \frac{1}{2} |\mathbf{r} \times \mathbf{v}| dt, \quad \dot{A} = \frac{1}{2} |\mathbf{r} \times \mathbf{v}|.$$

The cross product points perpendicular to the orbital plane. Call its direction $\hat{\mathbf{n}}$. In vector form,

$$\dot{A} \hat{\mathbf{n}} = \frac{1}{2} \mathbf{r} \times \mathbf{v}.$$

The plane stays fixed, so $\hat{\mathbf{n}}$ is constant. Kepler tells us $\dot{A}$ is constant too.

Now differentiate. The left side gives zero; the right side needs the product rule:

$$0 = \ddot{A} \hat{n} = \frac{1}{2} (\mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a}).$$

The first term is zero: a vector crossed with itself gives zero. That leaves just

$$\mathbf{r} \times \mathbf{a} = \mathbf{0}.$$

So $\mathbf{a}$ lies along $\mathbf{r}$, in the same or opposite direction. Every change in velocity is radial.

Conclusion: all changes of velocity are radial. The interaction is central.

The orbit bends toward the Sun, so the acceleration points inward. The area law fixes the radial line; it does not by itself fix the sign or strength.

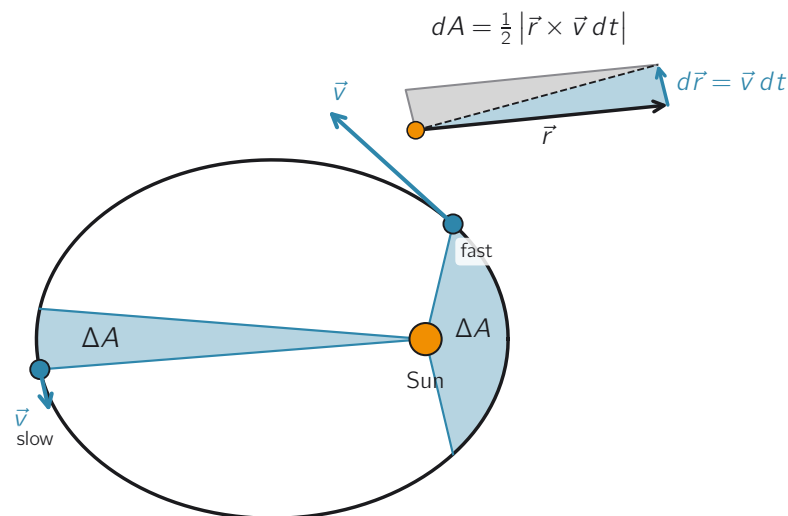


Figure 3: **Equal areas in equal times.** The two shaded sectors are swept in the same time interval Δt and have equal areas ΔA . Near the Sun the planet moves fast and the sector is short and fat; far away it moves slowly and the sector is long and thin. The inset enlarges one sliver: in a time dt the planet moves by $d\mathbf{r} = \mathbf{v} dt$ and sweeps a triangle of area $dA = \frac{1}{2} |\mathbf{r} \times \mathbf{v} dt|$.

Equal areas fix the line along which the acceleration acts, not how its strength depends on distance. The centered ellipse of [Section 1](#) also sweeps equal areas about its center, because $-\Omega^2 \mathbf{r}$ is parallel to $\mathbf{r}$. A planet's ellipse is different: its equal areas are swept about the Sun, at a focus, not about the center. Something beyond equal areas must decide how the pull depends on distance.

4.3 Distance dependence

What do the planets tell us? Larger orbits take longer to complete. The measured orbital sizes and periods follow Kepler's third law, $T^2 \propto a_{\text{orb}}^3$:

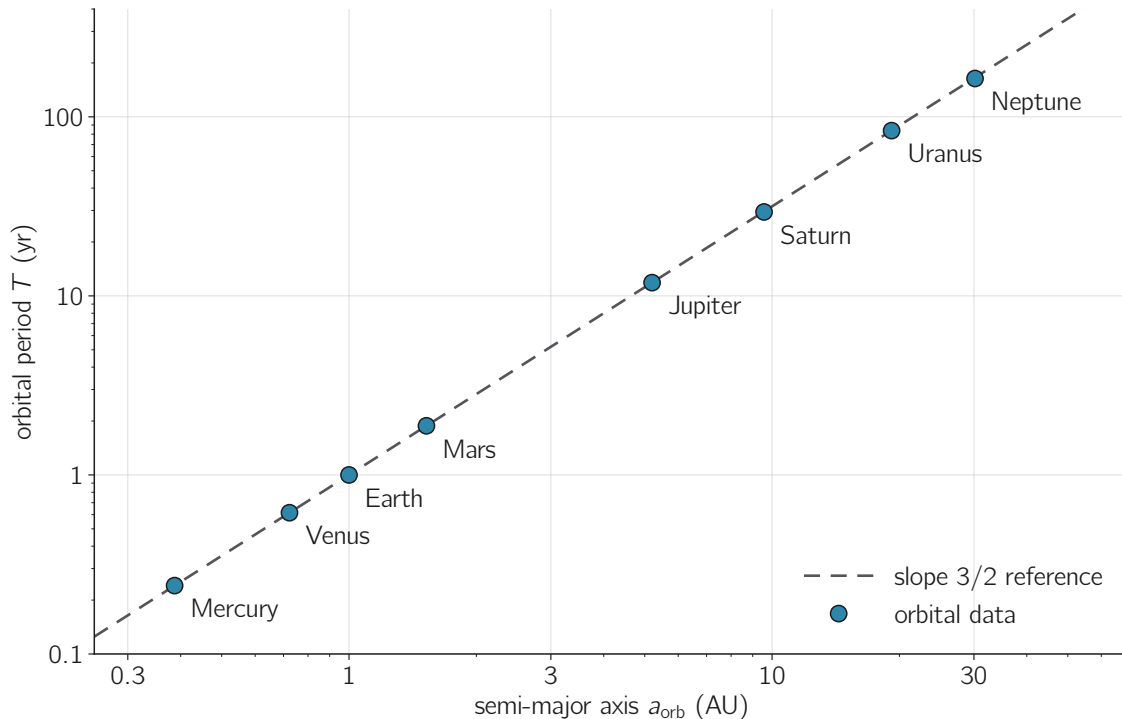


Figure 4: **What the planets show.** Published orbital parameters from the [NASA Planetary Fact Sheet](#). The dashed line has slope 3/2, corresponding to $T^2 \propto a_{\text{orb}}^3$, and passes through Earth's point. These are orbital sizes and periods, not accelerations calculated from a force law. No circular-orbit approximation is used.

Together with the elliptical shapes and the area law, this timing tells us how the inward acceleration depends on distance:

$$|\mathbf{a}| \propto \frac{1}{r^2}.$$

For the centered ellipse we had $|\mathbf{a}| = \Omega^2 r$; here the acceleration gets weaker farther away. Twice as far means one quarter of the acceleration. We will write the equation of motion in [Section 7](#).

5 Universal Gravitation

The Sun pulls on the Earth. But why should the Sun be special? The law should not need to know that one body is the Sun: it should work for any pair of bodies. Nobody is special. The Earth pulls on the Sun too; the Earth and Moon pull on each other, and so do the Sun and Moon. Everything pulls on everything else. Newton's step is to make the pull of the Sun one case of a universal interaction.

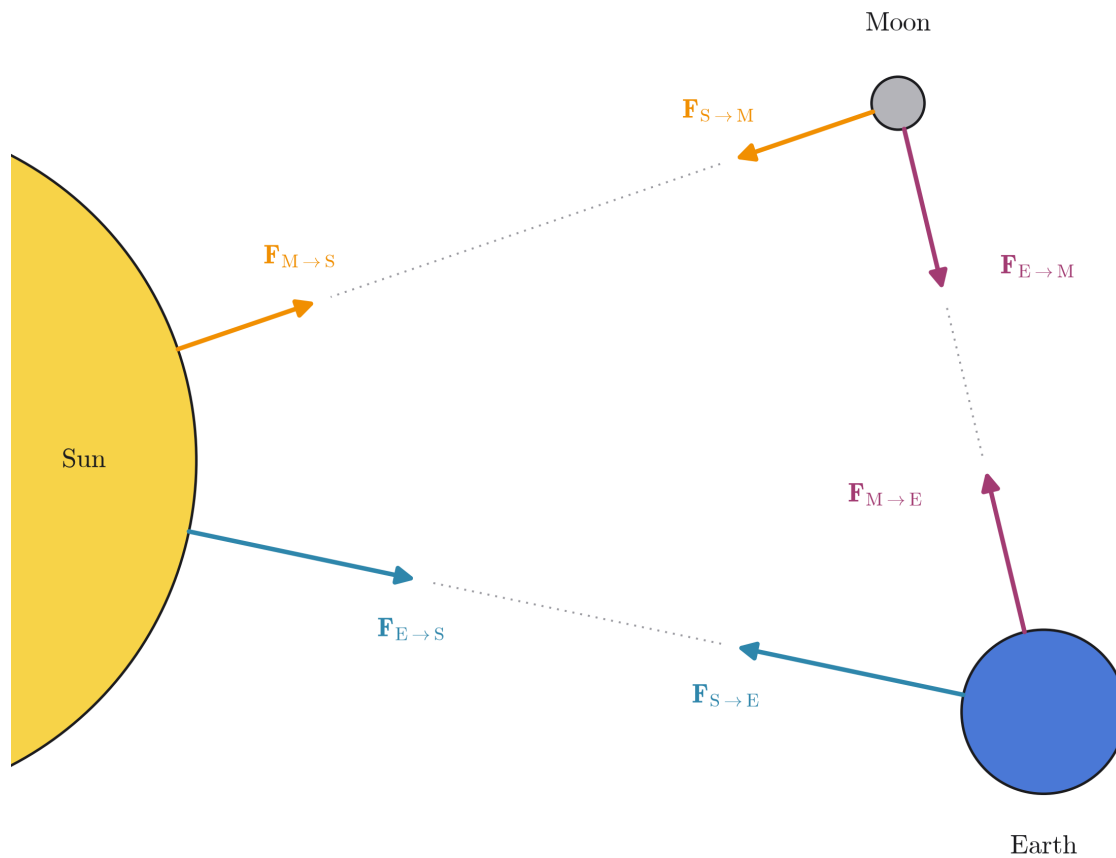


Figure 5: Each body feels a pull from each of the other two. The force $\mathbf{F}_{A \rightarrow B}$ is exerted by A on B : its arrow starts on B and points toward A . Matching colors identify the two forces within each pair. Positions and body sizes are schematic; only part of the Sun is shown. Each pair uses its own arrow-length scale so that all six forces are visible; lengths of different colors cannot be compared.

Newton asked whether the interaction responsible for an apple's fall could also account for the Moon's orbit. At the Moon's distance, gravity is weaker; its sideways motion carries it around the Earth as it falls.

An orbit is continuous free fall. In a circular orbit the body is always falling and always missing the Earth. Without gravity it would move along the straight tangent; gravity bends its path inward while the curved surface of the Earth falls away beneath it. At the circular-orbit speed the body stays at the same distance from the center, and so at the same height above a spherical Earth. This is free fall, not the absence of gravity. We neglect air resistance. In an elliptical orbit the distance changes.

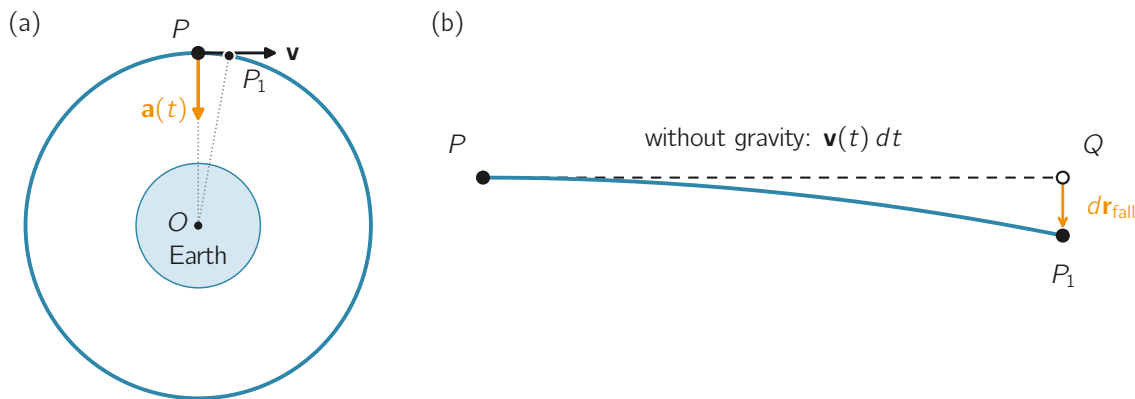


Figure 6: Always falling, always missing. (a) At P the instantaneous acceleration points toward Earth's center O ; the velocity is tangent to the circular orbit. (b) One short step, enlarged: without gravity the body would reach Q ; gravity displaces it toward P_1 . Measured from the gravity-free endpoint, not from P , this displacement is $d\mathbf{r}_{\text{fall}} \approx \frac{1}{2}\mathbf{a}(t)(dt)^2$, the leading short-time approximation, nearly parallel to the acceleration at P . This is the deflection from the straight-line path, not the total displacement $d\mathbf{r} = \mathbf{v} dt$. Over a finite interval the acceleration turns, so the accumulated displacement need not point exactly toward O . Schematic; the panels use different scales.

5.1 The force law

Nobody is special, yet the Sun dominates the planets' motion. What makes its pull so important? There must be something about the body itself that sets the strength of the interaction. We associate with each body a *quantity of matter*, its **mass**. At the same separation, more massive bodies attract each other more strongly.

Whose mass should appear: the body doing the pulling, or the body being pulled? Both. Each body does both jobs, so there is no reason to single out one of them: both forces must depend on both masses in the same way, and exchanging the labels of the bodies must leave the strength of their interaction unchanged.

Newton's gravitational law makes this quantitative: the force is proportional to each mass. Double either mass at fixed separation and the force doubles; double both and it becomes four times as large. That gives the product $m_1 m_2$. The symmetry motivates treating both masses equally; the proportionality to each mass is a physical statement of the law.

Combine this with the inverse-square dependence on distance and the direction along the line joining the bodies. Take two point masses m_1 and m_2 , and let $\mathbf{r}_{12}$ be the displacement from body 1 to body 2. Then

$$\mathbf{F}_{1 \rightarrow 2} = -G \frac{m_1 m_2}{r_{12}^2} \hat{\mathbf{r}}_{12},$$

Here $r_{12} = |\mathbf{r}_{12}|$, and $\hat{\mathbf{r}}_{12} = \mathbf{r}_{12}/r_{12}$ is the unit vector pointing from 1 to 2. The minus sign makes the force on body 2 point *back toward* body 1. For positive masses, Newtonian gravity is attractive.

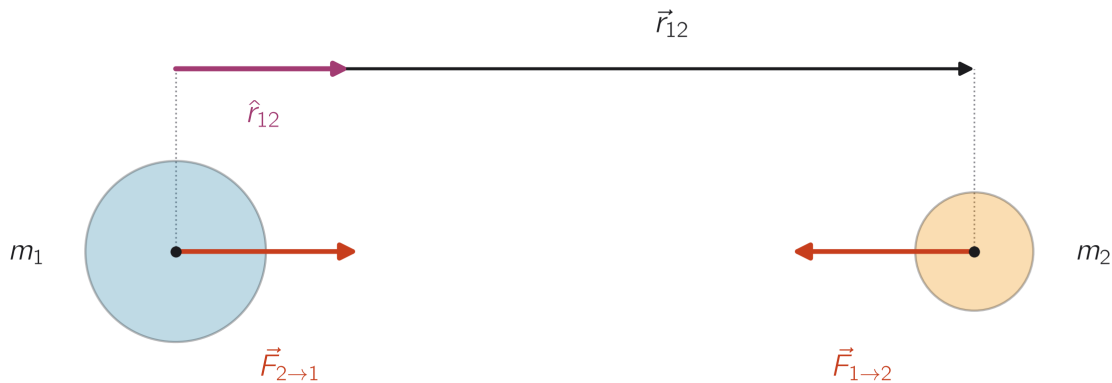


Figure 7: The displacement $\mathbf{r}_{12}$ points from body 1 to body 2. The dimensionless unit vector $\hat{\mathbf{r}}_{12}$ specifies that direction; its drawn length does not represent a physical distance. The force arrows point toward the other body and have equal lengths because the force magnitudes are equal.

The remaining proportionality factor is the **universal gravitational constant** G , the same for every pair of bodies whatever their masses, separation or identities.

Guess the units of G

Force is measured in newtons (N), mass in kilograms, and distance in meters. From $F = Gm_1m_2/r^2$, what units must G have?

Solve the force law for G and carry the units:

$$G = \frac{Fr^2}{m_1m_2}, \quad [G] = \frac{\text{N m}^2}{\text{kg}^2}.$$

Dimensional reasoning fixes the units, not the value. Measurement gives

$$G \approx 6.674 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2 = 6.674 \times 10^{-11} \text{ m}^3/(\text{kg} \cdot \text{s}^2).$$

The two forms agree because the newton is $1 \text{ N} = 1 \text{ kg} \cdot \text{m/s}^2$ (Section 6). The numerical value of a dimensional constant depends on the units; a force shows the physical scale. Two 1 kg point masses 1 m apart attract with only $6.7 \times 10^{-11} \text{ N}$. Gravity dominates large astronomical systems because mass accumulates, whereas ordinary matter is nearly electrically neutral.

“Universal” means that the same constant G describes terrestrial and celestial gravitational attraction. Newton’s inverse-square law is a weak-field, slow-motion approximation to gravity; its predictions are highly accurate for the situations considered here.

6 Newton’s Laws of Motion

6.1 From changing velocity to quantity of motion

The gravitational law describes one particular force. Return to the general statement: a force is what changes a velocity. For one fixed body, experiments relate the net force to the rate of change of velocity:

$$\mathbf{F}_{\text{net}} \propto \frac{d\mathbf{v}}{dt}.$$

Check the units: $[\mathbf{F}] = \text{N} = \text{kg m/s}^2$, whereas $[d\mathbf{v}/dt] = \text{m/s}^2$. The missing coefficient has units of mass. Units suggest what to look for; experiment establishes the relation.

The coefficient depends on the body. Push an empty cart and a loaded cart on a level, frictionless track with the same calibrated net force: the loaded cart changes its velocity more slowly.

That coefficient is the inertial mass m : the quantity of matter, measured by how strongly the body resists changes of velocity, not motion itself, and not only when starting from rest. Unlike the universal constant G , m belongs to the body. In gravitation, mass measured the strength of attraction; here it measures the inertial response. **That the two agree is an empirical fact.**

Newton’s words

Before stating his laws in the *Principia* (1687), Newton defines *quantity of matter* from density and volume together, and calls this quantity *mass*. He then defines *quantity of motion*. A literal translation of Definition II reads:

The quantity of motion is its measure arising from velocity and quantity of matter jointly.

He illustrates the combination: doubling the mass at the same velocity doubles the quantity of motion; doubling both makes it four times as large. See [Newton’s original definitions](#) ↗.

We are led to write $\mathbf{F}_{\text{net}} = m\mathbf{a}$, but Newton’s wording keeps mass and velocity together: his *quantity of motion* is $m\mathbf{v}$. In the *Principia* (1687), Law II begins, in translation: “The change of motion is proportional to the motive force impressed.” See [Newton’s original second law](#) ↗. He did not print the modern derivative notation used below.

Today the quantity of motion is called **momentum**, $\mathbf{p} = m\mathbf{v}$, with units $\text{kg} \cdot \text{m/s}$. For a body made of a fixed collection of matter, force is the rate of change of momentum:

$$\mathbf{F}_{\text{net}} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v}).$$

Here $\mathbf{v}$ is the body's center-of-mass velocity. For constant mass,

$$\frac{d}{dt}(m\mathbf{v}) = m \frac{d\mathbf{v}}{dt} = m\mathbf{a}.$$

Thus $\mathbf{F}_{\text{net}} = m\mathbf{a}$ is exactly the constant-mass form.

If the mass changes, as when a rocket ejects fuel, $d(m\mathbf{v})/dt$ also contains $\dot{m}\mathbf{v}$, and the momentum carried away by the ejected matter must be counted; applying the law to the remaining rocket alone leaves it out. Week 7 returns to this.

6.2 First law

The principle of inertia of [Section 3](#), that motion persists when impediments are removed, now has a compact form. With no net force,

$$\mathbf{F}_{\text{net}} = \mathbf{0} \implies \frac{d\mathbf{p}}{dt} = \mathbf{0} \implies \mathbf{p} = \text{constant}.$$

The momentum is a constant of motion; for fixed m , the velocity is constant. Newton stated this as his first law in the *Principia* (1687), for uniform motion in a straight line.

Newton's First Law. *A body subject to no net force moves with constant velocity. A body at rest remains at rest; a body in motion continues in motion in a straight line at constant speed.*

Newton's laws hold in an **inertial frame**, one in which a body with no interactions moves at constant velocity; in an accelerating or rotating frame, extra terms appear in the equation of motion.

6.3 Second law

What if several forces act on the body? Add them as vectors. Their sum tells us how its momentum changes:

$$\sum_i \mathbf{F}_i = \mathbf{F}_{\text{net}} = \frac{d\mathbf{p}}{dt}.$$

Here $\mathbf{p}$ is the momentum of the same body, with no matter entering or leaving it. For constant mass, $\mathbf{p} = m\mathbf{v}$ gives

$$\frac{d\mathbf{p}}{dt} = m \frac{d\mathbf{v}}{dt} = m\mathbf{a}.$$

Force therefore has units of momentum divided by time: $\text{kg} \cdot \text{m/s}^2$. This unit is called the **newton**, $1 \text{ N} = 1 \text{ kg} \cdot \text{m/s}^2$.

7 Orbits from Newton's Laws

The force law of [Section 5](#) and the second law of [Section 6](#) together give the equation of motion of a planet.

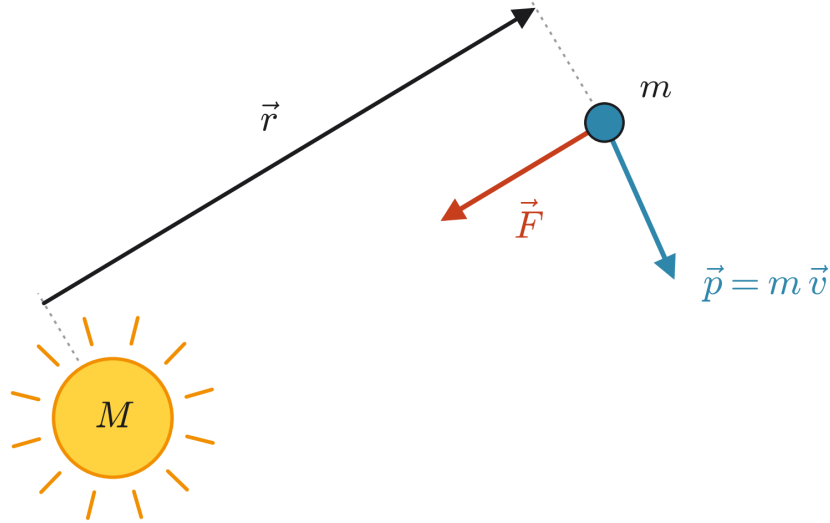


Figure 8: The position vector $\mathbf{r}$ points from the central mass M to the planet m ; it is drawn parallel to the line of centers with dotted guides for clarity. Gravity $\mathbf{F}$ points back toward M . The blue arrow is the quantity of motion $\mathbf{p} = m\mathbf{v}$, directed along the instantaneous velocity. Sizes and arrow lengths are schematic.

The orbital differential equation

For a planet with mass much smaller than a central mass M , treat the central body as fixed. Let $\mathbf{r}$ point outward from that body to the planet, with $r = |\mathbf{r}|$ and $\hat{\mathbf{r}} = \mathbf{r}/r$. The gravitational force on the planet is

$$\mathbf{F} = -\frac{GMm}{r^2} \hat{\mathbf{r}} = -\frac{GMm}{r^3} \mathbf{r}.$$

For a gravitational ellipse, this origin is the occupied focus, not the geometric center of the ellipse. We may choose another fixed origin, but the force must still use the separation vector from the Sun to the planet.

Write the planet's momentum as $\mathbf{p} = m\mathbf{v} = m\dot{\mathbf{r}}$. Its mass m stays constant, so

$$\frac{d\mathbf{p}}{dt} = m\ddot{\mathbf{r}} = \mathbf{F} = -\frac{GMm}{r^3} \mathbf{r}.$$

The planet's mass m cancels. With $\mu \equiv GM$, the constant for the chosen central body has units $[\mu] = \text{m}^3/\text{s}^2$.

For an object at position $\mathbf{r}$ under the gravitational attraction of a mass M fixed at the origin, the equation of motion is

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3} \mathbf{r}, \quad \mu = GM.$$

Here $r = |\mathbf{r}|$, and the minus sign points the acceleration toward the origin.

Compare with the rule of [Section 1](#), $\ddot{\mathbf{r}} = -\Omega^2 \mathbf{r}$. Both accelerations point opposite to $\mathbf{r}$, but the coefficient μ/r^3 changes with the distance. That one difference changes the orbits: bound

orbits become ellipses with the attracting body at a focus instead of the center. This single equation describes every orbit about the central body; the initial position and velocity select which one.

The Sun is not exactly fixed. The Earth pulls on it with a force of the same magnitude, so the Sun's acceleration is m/M times Earth's; holding it fixed is an approximation. Other bodies pull on the Earth as well, and the forces add. With Mars included, in an inertial frame,

$$\mathbf{F}_{\text{net}} = \mathbf{F}_{\text{Sun} \rightarrow \text{Earth}} + \mathbf{F}_{\text{Mars} \rightarrow \text{Earth}}, \quad \frac{|\mathbf{F}_{\text{Mars} \rightarrow \text{Earth}}|}{|\mathbf{F}_{\text{Sun} \rightarrow \text{Earth}}|} = \frac{m_{\text{Mars}}}{M_{\text{Sun}}} \left(\frac{r_{\text{Sun-Earth}}}{r_{\text{Mars-Earth}}} \right)^2.$$

Both mass and distance matter when deciding which pull dominates. Mars also pulls along a different line, so the geometry sets its net effect. A numerical calculation can include such small terms directly.

Separating radial and tangential motion

Return to the fixed-Sun approximation and neglect the other planets. The force points toward one center, so polar coordinates separate the change in distance from the motion around that center: $r(t)$ is the distance, and $\theta(t)$ locates the planet around the center, increasing in the direction of orbital motion. Because $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ turn as the planet moves, differentiating the position twice gives more than $\ddot{r} \hat{\mathbf{r}}$:

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}},$$

Gravity has only a radial component:

$$-\frac{\mu}{r^3}\mathbf{r} = -\frac{\mu}{r^2}\hat{\mathbf{r}} + 0\hat{\boldsymbol{\theta}}.$$

Matching the coefficients of $\hat{\mathbf{r}}$ and of $\hat{\boldsymbol{\theta}}$ gives the radial and tangential equations:

$$\underbrace{\ddot{r} - r\dot{\theta}^2}_{\text{radial acceleration}} = -\frac{\mu}{r^2}, \quad \underbrace{r\ddot{\theta} + 2\dot{r}\dot{\theta}}_{\text{tangential acceleration}} = 0.$$

These are two coupled equations for two unknown functions, $r(t)$ and $\theta(t)$. The term $-r\dot{\theta}^2$ accounts for the turning of the radial direction.

Look first at the tangential equation. We have already met this condition! In [Section 4.2](#), the area law told us that the acceleration is radial. Here it is again: the tangential acceleration is zero. Let us see the same area law come out of the equation of motion. That does not say that $\dot{\theta}$ is constant: the two terms on the left can cancel while both r and $\dot{\theta}$ change. Multiply the equation by r to make a product derivative appear:

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad \Longleftrightarrow \quad \frac{d}{dt}(r^2\dot{\theta}) = 0.$$

Check the equivalence with the product rule:

$$\frac{d}{dt}(r^2\dot{\theta}) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = r(r\ddot{\theta} + 2\dot{r}\dot{\theta}),$$

For $r > 0$, then, $r^2\dot{\theta}$ has zero time derivative. This says more than the acceleration at one instant: a combination of position and motion keeps the same value throughout the orbit, a **constant of motion**. Denote its value by h :

$$r^2\dot{\theta} = h = \text{constant}.$$

Look, the area law is already here! This is the same conserved quantity from our swept-area argument: $h = 2\dot{A}$, twice the areal rate. Its units are m^2/s . To see the connection, separate the velocity into its radial and tangential parts:

$$\mathbf{v} = \dot{r} \hat{\mathbf{r}} + r\dot{\theta} \hat{\boldsymbol{\theta}}, \quad \mathbf{v}_\theta = r\dot{\theta} \hat{\boldsymbol{\theta}}.$$

The tangential component is $v_\theta = r\dot{\theta}$. It is not the full speed when r also changes. Thus our constant is

$$h = r^2\dot{\theta} = r v_\theta.$$

Now put back the direction. Let $\hat{\mathbf{n}} = \hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}$ be the fixed unit vector perpendicular to the orbital plane. Then

$$h\hat{\mathbf{n}} = r(r\dot{\theta})\hat{\mathbf{n}} = (r\dot{\theta}) \times (r\dot{\theta} \hat{\boldsymbol{\theta}}) = \mathbf{r} \times \mathbf{v}_\theta.$$

The radial velocity contributes nothing to the cross product, because it is parallel to $\mathbf{r}$:

$$\mathbf{r} \times (\dot{r} \hat{\mathbf{r}}) = \mathbf{0}, \quad \mathbf{r} \times \mathbf{v} = \mathbf{r} \times \mathbf{v}_\theta = h\hat{\mathbf{n}}.$$

This is the combination from our area argument. With θ increasing in the direction of motion,

$$\dot{A} = \frac{1}{2} |\mathbf{r} \times \mathbf{v}| = \frac{h}{2}.$$

So these are the same statement: equal areas in equal times mean that the tangential acceleration is zero, and zero tangential acceleration means equal areas in equal times. For this motion in a fixed plane, with $r > 0$,

$$\dot{A} = \text{constant} \iff h = r^2\dot{\theta} = \text{constant} \iff a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.$$

The initial conditions set $h = r(0)^2\dot{\theta}(0)$. Closer to the center, both $v_\theta = h/r$ and $\dot{\theta} = h/r^2$ are larger. This follows from zero tangential force alone, before using the inverse-square dependence of the radial force.

Now return to the radial equation and eliminate $\dot{\theta} = h/r^2$ from

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}, \quad r^2\dot{\theta} = h,$$

Substituting gives $r\dot{\theta}^2 = r(h/r^2)^2 = h^2/r^3$. Rearranging,

$$\ddot{r} = \frac{h^2}{r^3} - \frac{\mu}{r^2}.$$

The term h^2/r^3 comes from the changing radial direction as the planet moves around the center; $-\mu/r^2$ comes from gravity. The constant h carries the angular initial condition.

⚠ Warning: these two second derivatives are different

The vector $\mathbf{r}$ tells us position; $r = |\mathbf{r}|$ tells us distance. Their second derivatives answer different questions:

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^2}\hat{\mathbf{r}}, \quad \ddot{r} = -\frac{\mu}{r^2} + \frac{h^2}{r^3}.$$

Why the extra term? The direction $\hat{\mathbf{r}}$ turns as the planet moves. The radial component of acceleration is $a_r = \ddot{r} - v_\theta^2/r$, not just $\ddot{r}$. The extra term accounts for this turning; it is not an outward force.

Check a circle. The distance stays fixed, while the velocity turns: $a_r = -v_\theta^2/R = -\mu/R^2$. So

$$\ddot{r} = 0, \quad \ddot{\mathbf{r}} = -\frac{\mu}{R^2}\hat{\mathbf{r}}.$$

Constant distance means $\ddot{r} = 0$, not $a_r = 0$: the radial acceleration is still inward.

7.1 Kepler's third law: circles and ellipses

The simplest solution is a circular orbit: $r(t) = R$, so $\dot{r} = \ddot{r} = 0$, and the radial equation becomes

$$0 = \frac{h^2}{R^3} - \frac{\mu}{R^2} \implies h^2 = \mu R.$$

With $h = R^2\dot{\theta}$,

$$R^4\dot{\theta}^2 = \mu R, \quad \dot{\theta}^2 = \frac{\mu}{R^3}.$$

With θ increasing along the motion, the angular speed is constant:

$$\omega \equiv \dot{\theta} = \sqrt{\frac{\mu}{R^3}}, \quad \theta(t) = \theta_0 + \omega t.$$

The radius stays fixed and the angle increases uniformly. In one period T the angle increases by 2π :

$$\omega T = 2\pi, \quad \omega = \frac{2\pi}{T}.$$

Substitute this into $\omega^2 = \mu/R^3$:

$$\frac{4\pi^2}{T^2} = \frac{\mu}{R^3} \implies T^2 = \frac{4\pi^2}{\mu} R^3 = \frac{4\pi^2}{GM} R^3.$$

This is Kepler's third law for circular orbits, with consistent units: $[R^3/\mu] = \text{s}^2$. The ratio T^2/R^3 is shared by orbits around the same central mass; it compares different orbits, unlike a quantity conserved along one orbit. The law holds for ellipses too, with the semi-major axis in place of R : $T^2 = 4\pi^2 a_{\text{orb}}^3/\mu$. We continue to neglect the planet's mass relative to M and the attractions of other bodies.

∴ Constructing the motion numerically

When guessing a solution no longer works, keep a table of numbers instead of a function. Each row holds t , $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{a}$. Steps are finite, Δt , not dt .

Row 0, $t = 0$. Start at the closest or farthest point, where the velocity is purely tangential:

$$\mathbf{r}_0 = r_0 \hat{\mathbf{r}}(0), \quad \mathbf{v}_0 = v_0 \hat{\boldsymbol{\theta}}(0), \quad \mathbf{a}_0 = \mathbf{a}(0) = -\frac{\mu}{r_0^2} \hat{\mathbf{r}}(0).$$

Use this vector acceleration, not $\ddot{r} \hat{\mathbf{r}}$ (warning above).

Row 1, $t = \Delta t$. Hold $\mathbf{v}_0$ and $\mathbf{a}_0$ fixed for one step, then recompute $\mathbf{a}$:

$$\mathbf{r}_1 = \mathbf{r}_0 + \mathbf{v}_0 \Delta t, \quad \mathbf{v}_1 = \mathbf{v}_0 + \mathbf{a}_0 \Delta t, \quad \mathbf{a}_1 = -\frac{\mu}{|\mathbf{r}_1|^3} \mathbf{r}_1.$$

Row 2, $t = 2\Delta t$, repeats this from row 1, and so on. With $t_n = n\Delta t$,

$$\mathbf{r}_{n+1} = \mathbf{r}_n + \mathbf{v}_n \Delta t, \quad \mathbf{v}_{n+1} = \mathbf{v}_n + \mathbf{a}_n \Delta t, \quad \mathbf{a}_{n+1} = -\frac{\mu}{|\mathbf{r}_{n+1}|^3} \mathbf{r}_{n+1}.$$

This is forward Euler: $\mathbf{r}_1 \approx \mathbf{r}(\Delta t)$ and $\mathbf{v}_1 \approx \mathbf{v}(\Delta t)$ are estimates of the continuous motion.

Earth: a numerical orbit and its table

Earth's orbit is nearly circular ($e = 0.0167$). With a fixed Sun at the origin and $\mu = 1.3271244004 \times 10^{11} \text{ km}^3/\text{s}^2$, start Earth at its closest point:

$$\mathbf{r}(0) = (147.099586 \times 10^6, 0) \text{ km}, \quad \mathbf{v}(0) = (0, 30.286320) \text{ km/s}.$$

The two vectors are perpendicular, so $h = |\mathbf{r}(0)| |\mathbf{v}(0)| \approx 4.4551 \times 10^{15} \text{ m}^2/\text{s}$ and the areal rate is $\dot{A} = h/2 \approx 2.2276 \times 10^{15} \text{ m}^2/\text{s}$. Every second the Sun–Earth line sweeps out about 4.4 times Earth's entire surface, oceans included, $4\pi R_\oplus^2 \approx 5.10 \times 10^{14} \text{ m}^2$.

The forward Euler calculation uses 200 000 steps over one model period, $T = 365.256898 \text{ days}$, so $\Delta t = 157.790980 \text{ s}$. The table shows the first four updates and selected later times.

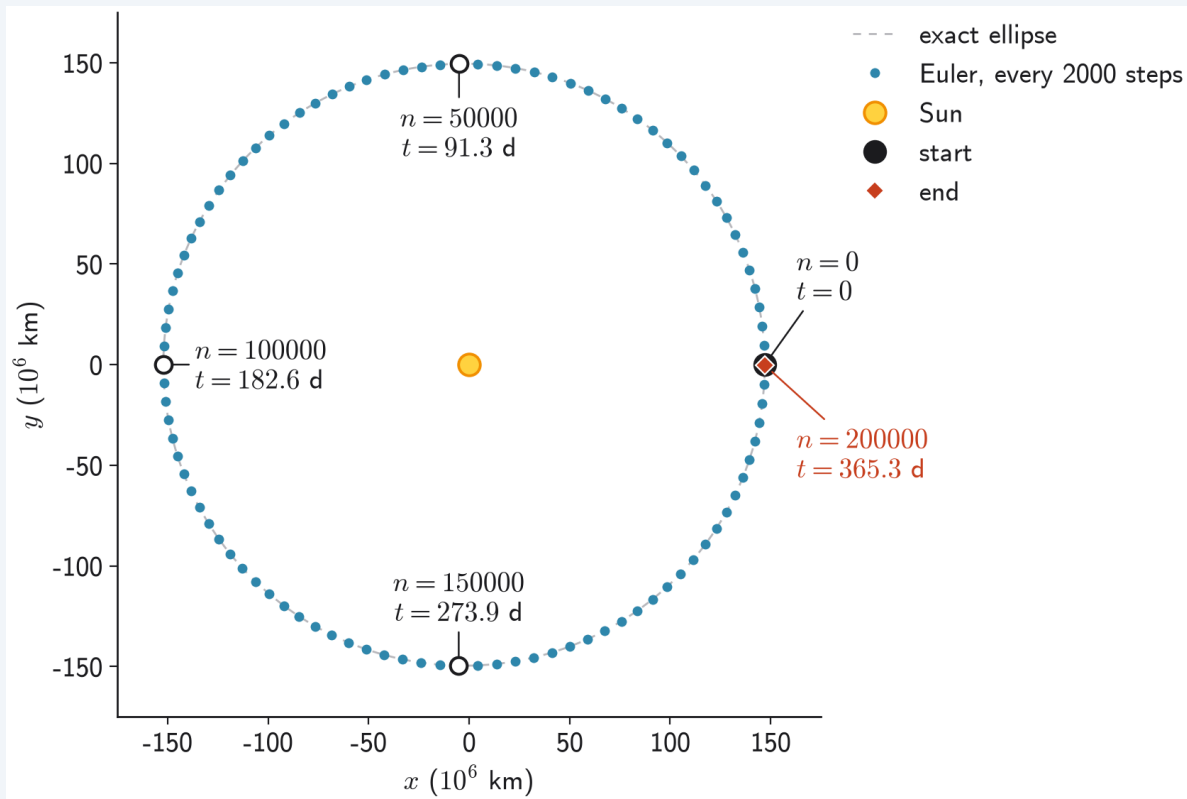


Figure 9: Building an orbit one step at a time. Blue dots: every 2000th forward Euler step, with labels matching the table. Dashed curve: the exact ellipse for these initial conditions. Start and end nearly overlap. Equal axis scales; Earth's small eccentricity is not exaggerated. Positions x, y are in million km; velocities v_x, v_y in km/s; accelerations a_x, a_y in mm/s^2 .

n	t (days)	x	y	v_x	v_y	a_x	a_y
0	0.000000	147.099586	0.000000	0.000000	30.286320	-6.133223	0.000000
1	0.001826	147.099586	0.004779	-0.000968	30.286320	-6.133223	-0.000199
2	0.003653	147.099586	0.009558	-0.001936	30.286320	-6.133223	-0.000399
3	0.005479	147.099586	0.014337	-0.002903	30.286320	-6.133223	-0.000598
4	0.007305	147.099585	0.019116	-0.003871	30.286320	-6.133223	-0.000797
50000	91.314225	-4.992854	149.564290	-29.772220	-0.494938	0.197720	-5.922843
100000	182.628449	-152.126563	0.042120	-0.011055	-29.288405	5.734578	-0.001588
150000	273.942674	-5.142006	-149.604690	29.765430	-0.526998	0.203441	5.919051
199999	365.255072	147.156680	-0.287983	0.058283	30.280431	-6.128429	0.011993
200000	365.256898	147.156689	-0.283205	0.057316	30.280433	-6.128430	0.011794

The endpoint error is about 289 000 km (0.196% of the starting distance); doubling the steps reduces it to about 144 000 km. This is numerical error: the ideal ellipse closes. The average-velocity position update, $\mathbf{r}_{n+1} = \mathbf{r}_n + \frac{1}{2}(\mathbf{v}_n + \mathbf{v}_{n+1})\Delta t$, is exact for constant acceleration.

Sources: [NASA Earth Fact Sheet](#); [JPL astrodynamic parameters](#).