

Mass, Springs and Friction

Week 05

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Readings

1. **Kleppner & Kolenkow, §2.9 and §§3.4–3.6:** *Applying Newton's laws; phenomenological forces.* Springs, contact forces and friction.
2. **Cole, Ch. 6:** Free-body diagrams and applications of Newton's second law.
3. **Feynman Lectures I, Ch. 12:** *Characteristics of Force.* Read §§12-1–12-2 on the meaning of force and the empirical laws of friction. [Read online ↗](#)

Overview

What does mass measure? It enters both the gravitational force and the response to a force. In free fall these two roles cancel, so the acceleration does not tell us the mass. A spring gives us a way to measure it: on the same spring, different masses oscillate with different periods. We then introduce damping and contact forces, and use Newton's second law to determine when a body stays at rest and when it accelerates.

1 Recap: Force and Planetary Motion

Newton's second law relates the net force on a body to the rate of change of its momentum $\mathbf{p}$:

$$\mathbf{F}_{\text{net}} = \frac{d\mathbf{p}}{dt}, \quad \mathbf{p} = m\mathbf{v}.$$

For the planets, the force is gravity. A body of mass M attracts a body of mass m at distance r with

$$\mathbf{F} = -\frac{GmM}{r^2} \hat{\mathbf{r}},$$

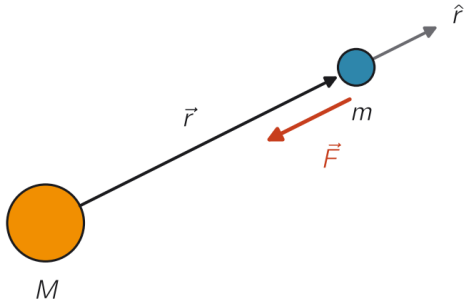
where $\hat{\mathbf{r}}$ points from M to m . The force lies along the line connecting the two bodies, points inward, and decays as $1/r^2$.

Because the gravitational acceleration is collinear with $\mathbf{r}$, pointing opposite to it for attractive gravity, we have $\mathbf{r} \times \mathbf{a} = 0$, and

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \dot{\mathbf{r}} \times \mathbf{v} + \mathbf{r} \times \dot{\mathbf{v}} = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a} = 0.$$

So $\mathbf{r} \times \mathbf{v}$ is a constant of the motion. In a time dt the planet moves by $\mathbf{v} dt$, and the thin triangle with sides $\mathbf{r}$ and $\mathbf{v} dt$ has area $dA = \frac{1}{2}|\mathbf{r} \times \mathbf{v}| dt$. The areal rate $\frac{1}{2}|\mathbf{r} \times \mathbf{v}|$ is therefore constant: Kepler's second law.

(a)



(b)

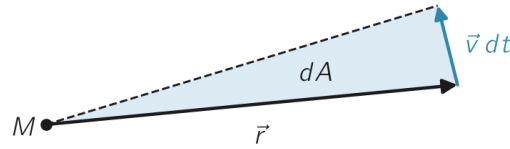


Figure 1: (a) Gravity on m points along the line to M , opposite to the outward unit vector $\hat{\mathbf{r}}$. (b) In a time dt the planet moves by $\mathbf{v} dt$; the shaded triangle with sides $\mathbf{r}$ and $\mathbf{v} dt$ has area $dA = \frac{1}{2}|\mathbf{r} \times \mathbf{v}| dt$. Sizes are schematic.

We now leave the planets and look at the two sides of $\mathbf{F}_{\text{net}} = d(m\mathbf{v})/dt$. The left side is the net force, the sum of every force acting on the body. Each of those forces needs its own force law, a model of the interaction that says who acts on the body and how. The right side says how the body's motion changes in response, and that depends on its mass.

2 Two Roles for Mass

2.1 Gravitational and inertial mass

Mass has entered our equations in two different ways. In the law of gravitation it measures how strongly a body is pulled. In the second law it sets how much acceleration a given net force produces: it measures the body's inertia. Nothing in the logic of the equations forces these to be the same quantity, so give them separate names. Let m_g be the **gravitational mass** of the falling body, m_i its **inertial mass**, and M_g the gravitational mass of the attracting body. Then

$$-\frac{GM_g m_g}{r^2} \hat{\mathbf{r}} = m_i \frac{d\mathbf{v}}{dt}, \quad \text{so} \quad \mathbf{a} = -\frac{m_g}{m_i} \frac{GM_g}{r^2} \hat{\mathbf{r}}.$$

If the ratio m_g/m_i differed from one body to another, two bodies released at the same place with the same initial velocity would accelerate differently. Experiment finds the opposite: at the same location, with the same initial conditions and with air resistance removed, test bodies fall with the same acceleration regardless of their mass or what they are made of. This is the

universality of free fall, also called the **weak equivalence principle**. The ratio m_g/m_i is therefore the same for all bodies, and choosing the units of m_g so that the ratio is one gives $m_g = m_i$ and

$$\mathbf{a} = -\frac{GM_g}{r^2} \hat{\mathbf{r}}.$$

The falling body's mass has cancelled. The equality $m_g = m_i$ is an empirical fact together with a conventional normalization; algebra alone does not require it. From here on we write a single m .

⊙ **Demonstration: a feather and a piece of metal**

A sealed tube holding a piece of metal and a feather tests this prediction. With the air pumped out, release both bodies from rest at the same height, so that they have the same initial conditions and gravity is the only appreciable force. If m_g/m_i is the same for both, they have the same acceleration and reach the bottom together, although the metal is far heavier. They do, as far as the eye can tell; a demonstration of this kind cannot measure m_g/m_i to high precision. Letting air back into the tube changes the experiment. Air resistance now acts on each body, and the feather, with a large surface for its small mass, falls far more slowly. The difference comes from the drag of the air, not from gravity.

Watch: Feather and Coin (Weber State University) ↗. Compare the fall with air in the tube and after the air is removed.

2.2 Gravity near Earth's surface

The law that governs the planets also governs a ball near the ground. Take $\hat{\mathbf{r}}$ pointing outward from Earth's center. For a ball of mass m at height h above the surface, $r = R_E + h$, where R_E is Earth's radius and M_E its mass. For heights small compared with Earth's radius, $h \ll R_E$, we have $r \approx R_E$ and

$$\mathbf{F} = -\frac{GmM_E}{r^2} \hat{\mathbf{r}} \approx -mg \hat{\mathbf{r}}, \quad g \equiv \frac{GM_E}{R_E^2}.$$

The minus sign makes the force point down, toward the center. Substituting the measured values of G , M_E and R_E gives $g \approx 9.8 \text{ m/s}^2$; for estimates we often round to $g \approx 10 \text{ m/s}^2$. Every ball near the surface, heavy or light, has the same downward acceleration g when air resistance is negligible.

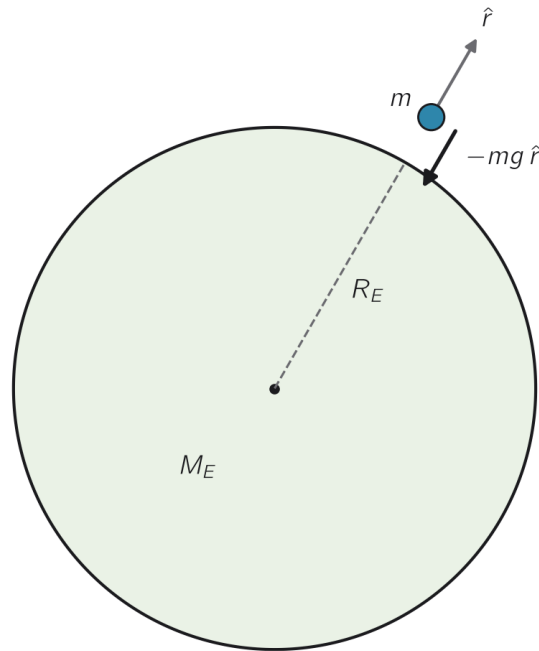


Figure 2: A small ball of mass m just above Earth's surface. Its weight $-mg \hat{r}$ points toward Earth's center, opposite to the outward $\hat{r}$. The ball and its height are drawn much larger than to scale.

① From falling bodies to the equivalence principle

In *Two New Sciences* (1638), Galileo's spokesman Salviati argues that, with the resistance of the medium removed, bodies of different weight would fall with the same speed. In the *Principia* (first published 1687), Book III, Proposition VI, Newton reports pendulums of equal length with bobs of different materials, including gold, glass, wood and wheat, swinging with equal periods: weight is proportional to quantity of matter. Beginning in 1907, Einstein extended the idea from falling bodies to all local non-gravitational physics: in a small, freely falling laboratory, where tidal effects are negligible, such physics proceeds as in a laboratory free of gravity. This Einstein equivalence principle is a stronger statement than the weak equivalence principle, which concerns only the motion of test bodies.

Sources: Galileo, *Two New Sciences*; Newton, *Principia* (1846 translation); Einstein's 1907 starting point; Einstein and the equivalence principle.

3 Fundamental and Effective Forces

Newton's second law needs a force law as input. Some force laws are **fundamental**. Gravity is one:

$$\mathbf{F} = -\frac{GmM}{r^2} \hat{r}.$$

The electric force between charges q and Q is another:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \hat{\mathbf{r}},$$

with $\hat{\mathbf{r}}$ pointing from Q to q and ϵ_0 a constant of nature; like charges repel. The bodies interact because they carry a property: mass for gravity, electric charge for the electric force. Each property is tied to a specific interaction: it tells us how the force acts on the body, giving the quantity its physical meaning. The link between a fundamental interaction and a property of matter is itself a deep and subtle fact. Asking why these properties and interactions are linked is one way we advance our understanding of physical laws. The bodies do not need to touch. The constants G and ϵ_0 are **universal constants set by nature**, not adjustable parameters we choose. Once we choose our units, experiment tells us their values. As far as we know, they are the same everywhere and do not change with time.

Now bring two bodies into contact. The normal force, friction, and the pull of a spring or rope are **phenomenological**, or effective, forces. They emerge from the combined effect of many microscopic interactions among the atoms, electrons and nuclei in the bodies. Rather than follow all those interactions, we describe their net effect with a simple force law. Its form and constants depend on the materials and conditions. A spring constant k , the drag coefficient c of [Section 5](#) and the friction coefficients μ_s and μ_k of [Section 6](#) belong to a particular spring, a particular medium, a particular pair of surfaces, and can change with temperature, wear or humidity. They are measured for the apparatus at hand.

- **Spring force:** a spring stretched or compressed by x from its relaxed length pushes back with $F_x = -kx$. The spring constant $k > 0$ is a property of the spring, with units $[k] = \text{N/m} = \text{kg/s}^2$.
- **Normal force:** objects do not fall through the floor. While two bodies stay in contact, a surface pushes perpendicular to itself with whatever force is needed to prevent penetration, up to what the material can withstand.
- **Tension:** an ideal rope does not stretch. A taut rope pulls along itself with whatever force keeps its length fixed, up to its breaking strength.
- **Static friction:** surfaces in contact resist the start of relative sliding, up to a limit.
- **Kinetic friction:** once surfaces slide, friction opposes the actual relative sliding. It does not in general cancel the applied force.

For a rigid contact, the normal force and static friction are found from the second law together with the conditions of no penetration and, while friction can prevent it, no sliding. A block at rest on a horizontal table keeps its momentum constant, $d\mathbf{p}/dt = 0$. If gravity and the normal force are its only vertical forces, the table must push up exactly as hard as gravity pulls down. Balanced forces mean zero change in momentum: the momentum remains constant, and it need not be zero. A body moving with constant momentum also has zero net force. Seeing a body move does not tell us that the forces on it are unbalanced; only a change in its momentum does.

A force that resists motion also arises when a body moves through air or water. We meet a simple version of it in [Section 5](#).

4 Measuring Mass with a Spring

Free fall cannot measure mass, because the gravitational force and the inertial mass grow by the same factor. A spring's force does not: it depends on how far the spring is stretched, not on what is attached to it. The same stretch accelerates a light block more than a heavy one, so a heavy block should oscillate more slowly. Take a spring from the laboratory, fix one end to a wall, and attach a block of mass m to the other end. The block rests on a frictionless horizontal surface. Let x be the block's displacement from its equilibrium position, where the spring is relaxed and $x = 0$. The spring exerts

$$F_x = -kx, \quad k > 0.$$

This is a **restoring force**: for $x > 0$ it points in the $-x$ direction, and for $x < 0$ in the $+x$ direction, always back toward the relaxed length.

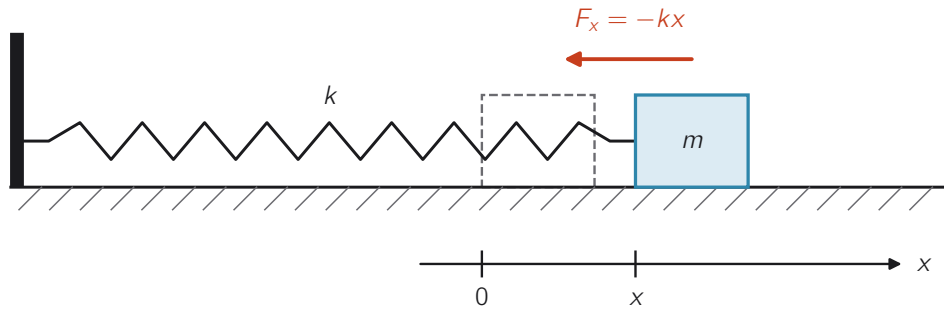


Figure 3: A block of mass m on a frictionless surface, attached to a spring of constant k . The dashed outline marks equilibrium, $x = 0$. At displacement x , the restoring force $F_x = -kx$ points back toward equilibrium.

Vertically, gravity and the normal force cancel. Along x , the spring is the only force. Start from Newton's second law:

$$F_x = \frac{dp_x}{dt}, \quad F_x = -kx.$$

The momentum is $p_x = mv_x = m\dot{x}$. For constant mass,

$$\frac{dp_x}{dt} = \frac{d}{dt}(m\dot{x}) = m\ddot{x}.$$

Putting the force and the change in momentum together gives

$$-kx = \frac{dp_x}{dt} = m\ddot{x},$$

so the equation of motion is

$$m\ddot{x} = -kx.$$

The mass here is the inertial mass.

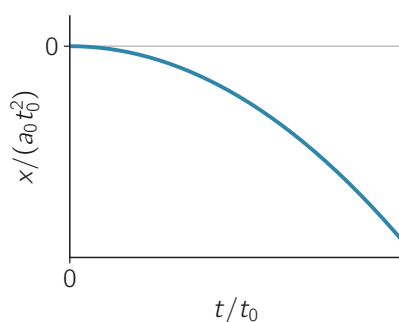
4.1 Solving the equation of motion

Starting displacement and period

The same block on the same spring is pulled out farther before it is released. Does the period get longer, shorter, or stay the same?

To solve the equation, guess a function, substitute it, and check whether the equation holds at every time.

1. Guess: constant acceleration.



Try

$$x(t) = -\frac{1}{2}a_0 t^2,$$

with constant a_0 . The sketch uses an arbitrary time scale t_0 . Does it solve the equation? Differentiate: $\dot{x} = -a_0 t$ and $\ddot{x} = -a_0$.

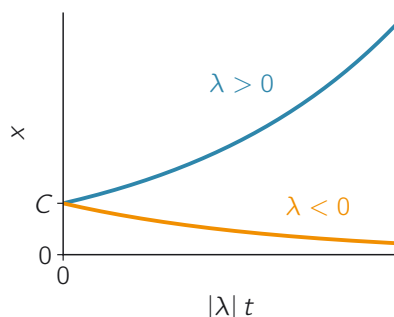
Substitution would require

$$-ma_0 = -k\left(-\frac{1}{2}a_0 t^2\right) = \frac{1}{2}ka_0 t^2.$$

For $a_0 \neq 0$, dividing by a_0 gives $-m = \frac{1}{2}kt^2$, which cannot hold for positive m and k .

× **This guess does not work.** Constant acceleration needs a constant force, but the spring force changes with x .

2. Guess: an exponential.



The equation asks for a function whose second derivative is proportional to the function itself. An exponential has this property, so try, with a constant C and a rate λ in units of 1/s,

$$x(t) = Ce^{\lambda t}.$$

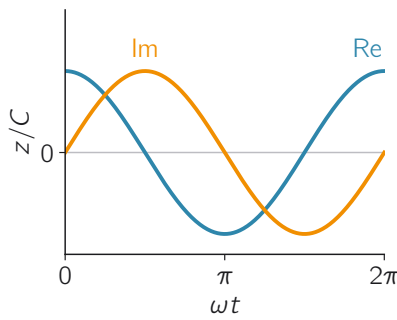
Differentiate to check:

$$\dot{x} = \lambda Ce^{\lambda t}, \quad \ddot{x} = \lambda^2 Ce^{\lambda t} = \lambda^2 x.$$

Substituting, $m\lambda^2 Ce^{\lambda t} = -kCe^{\lambda t}$, or $(m\lambda^2 + k)Ce^{\lambda t} = 0$. For $C \neq 0$ this holds at all times only if

$$m\lambda^2 + k = 0, \quad \lambda^2 = -\frac{k}{m}, \quad \lambda = \pm i\sqrt{\frac{k}{m}} \equiv \pm i\omega.$$

Here i is the imaginary unit, $i^2 = -1$. Since k and m are positive, no real λ works. A real exponent would make $e^{\lambda t}$ grow or decay steadily and never come back, which cannot describe a block pulled back toward $x = 0$ from either side. ✓ **The exponential guess works when we allow an imaginary exponent.** It then describes oscillation.



By Euler's identity,

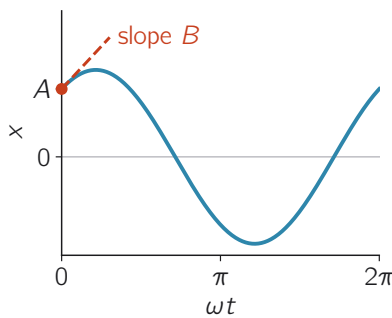
$$e^{\pm i\omega t} = \cos \omega t \pm i \sin \omega t,$$

whose real and imaginary parts are a cosine and a sine, shown on the left for $z = Ce^{i\omega t}$.

Both exponents give solutions, and because the equation is linear, so does any combination of them:

$$x(t) = C_+ e^{i\omega t} + C_- e^{-i\omega t} = (C_+ + C_-) \cos \omega t + i(C_+ - C_-) \sin \omega t.$$

3. Guess: cosine and sine.



We see that the previous guess is exactly a combination of cosine and sine. We could have guessed this same solution from the start:

$$x(t) = A \cos \omega t + B \sin \omega t.$$

The two forms agree when $A = C_+ + C_-$ and $B = i(C_+ - C_-)$. A position must be real. That requires $C_- = C_+^*$, the complex conjugate of C_+ , which makes A and B real. In practice we work directly with the real form, with real constants A and B ,

and check it by substitution.

Differentiating,

$$\dot{x} = -A\omega \sin \omega t + B\omega \cos \omega t, \quad \ddot{x} = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t = -\omega^2 x.$$

Substituting into the equation of motion, $-m\omega^2 x = -kx$ for all t , so

$$k = m\omega^2, \quad \omega = \sqrt{\frac{k}{m}}.$$

Euler's formula connects the two forms; they are the same solution.

The spring constant and the mass together set ω , the angular frequency: the phase ωt advances by 2π in one full cycle, which takes the **period**

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}.$$

✓ Units — Spring period

Since $[k] = \text{kg/s}^2$, m/k is in s^2 and T is in seconds.

The frequency, the number of cycles per second, is $1/T = \omega/(2\pi)$; the angular frequency ω is in rad/s, or simply 1/s.

The equation of motion has two time derivatives, and its general solution has two free constants, A and B . Two facts about the start of the motion fix them: the initial position and the initial velocity. Suppose that at $t = 0$ the block is at $x(0) = x_0$ and moving with $\dot{x}(0) = v_0$. Setting $t = 0$ in $x(t)$ and $\dot{x}(t)$ gives $x(0) = A$ and $\dot{x}(0) = B\omega$, so

$$A = x_0, \quad B = \frac{v_0}{\omega}, \quad x(t) = x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t.$$

The initial position alone does not determine B : because $\sin 0 = 0$, the sine term vanishes at $t = 0$ whatever B is. Only the initial velocity fixes it.

Now pull the block out to x_0 and release it from rest, so $v_0 = 0$ and $B = 0$:

$$x(t) = x_0 \cos \left(\sqrt{\frac{k}{m}} t \right).$$

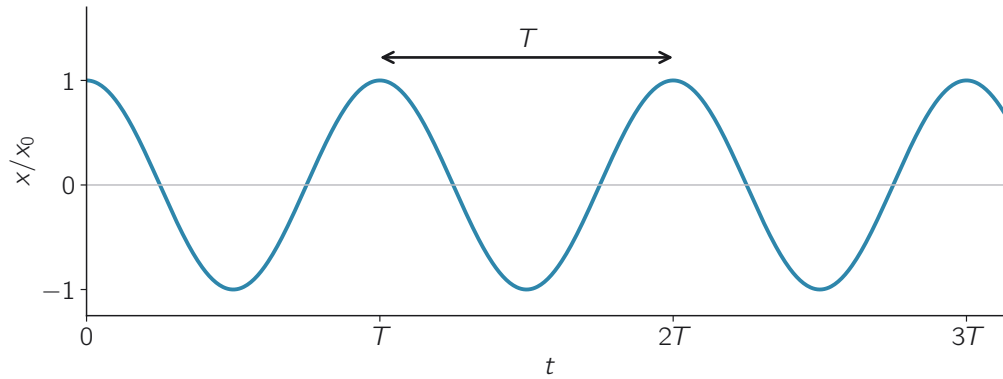


Figure 4: Released from rest at x_0 , the block oscillates as $x(t) = x_0 \cos \omega t$, with $\omega = \sqrt{k/m}$ and period $T = 2\pi/\omega$. The graph shows x/x_0 .

For $x_0 > 0$, the release point is a maximum of $x(t)$: a turning point, where the velocity is zero and the block reverses direction. It is not an inflection point. There the acceleration $\ddot{x} = -kx_0/m$ is largest in magnitude; the inflection points of the cosine are at $x = 0$, where the spring force vanishes and the speed is largest. The amplitude depends on how far we pulled the block. The period does not; it depends only on m and k .

4.2 Comparing periods

If we know k , measuring T gives the mass: $m = kT^2/(4\pi^2)$. Suppose instead that we do not know our spring well; it has never been calibrated. Take a second object whose mass m_{ref} is known, a calibrated reference mass, and attach it to the *same* spring, Figure 5(a). It oscillates with angular frequency $\omega_{\text{ref}} = \sqrt{k/m_{\text{ref}}}$ and period T_{ref} .

The ratio of the two periods is

$$\frac{T_{\text{ref}}}{T} = \frac{\omega}{\omega_{\text{ref}}} = \sqrt{\frac{k/m}{k/m_{\text{ref}}}} = \sqrt{\frac{m_{\text{ref}}}{m}}, \quad \text{so} \quad \boxed{m = m_{\text{ref}} \left(\frac{T}{T_{\text{ref}}} \right)^2}.$$

The unknown spring constant has cancelled. This comparison needs only the reference mass and a clock. In free fall the body's mass cancels; in the period of a spring it does not.

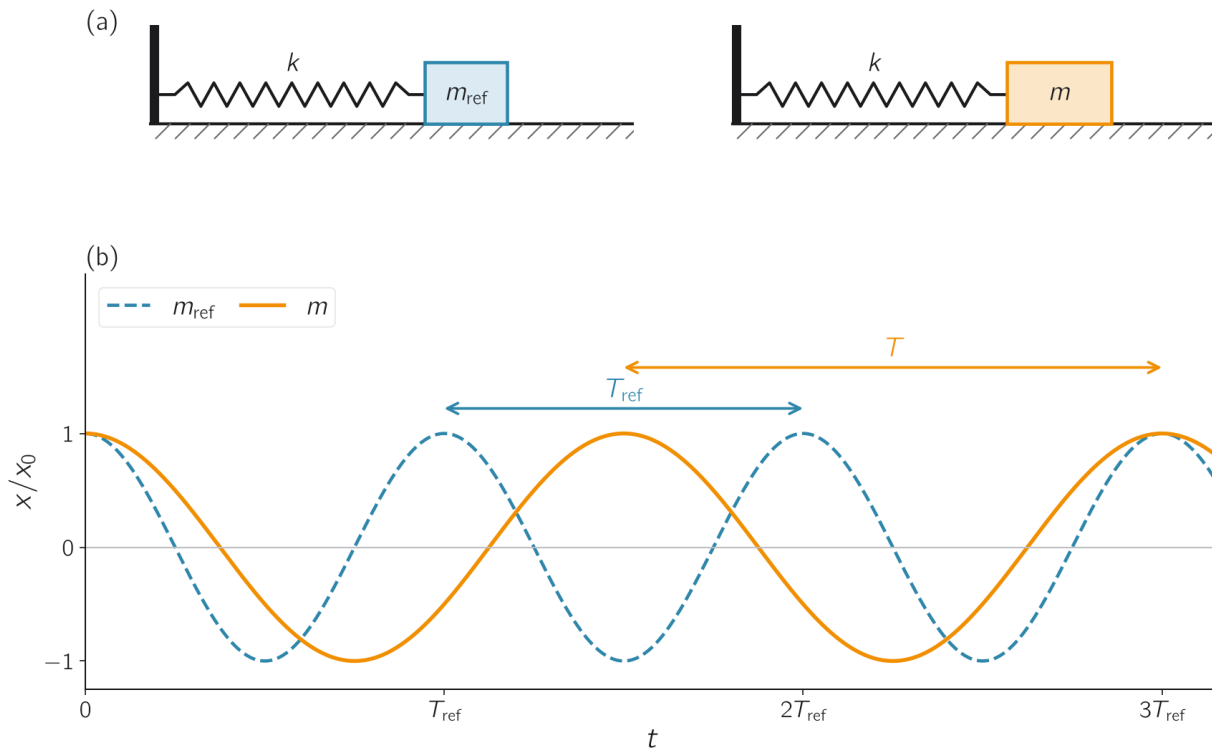


Figure 5: (a) The same spring of constant k carries the reference mass m_{ref} (left) or the unknown mass m (right). (b) Each is released from rest at x_0 : the reference mass m_{ref} (dashed) oscillates faster than the heavier unknown mass m (solid).

Example: comparing masses. If the unknown mass takes twice as long to complete a cycle on the same spring, then $T = 2T_{\text{ref}}$ and $m = 4m_{\text{ref}}$. Twice the period means four times the mass.

⊕ Demonstration: spring motion

Spring motion in the Differential Equations Lab ↗. Select only the position force $-kx$, keep k fixed, and vary m to compare the periods.

Mechanical quantities are built from three base dimensions, each measured with a familiar instrument: time T with a clock, length L with a ruler, as for the amplitude x_0 , and mass M by comparison with a reference mass, on a spring as above or on a balance. Their SI units are the second, the metre and the kilogram. Every mechanical unit in this course is a combination of them; for example, $1 \text{ N} = 1 \text{ kg m/s}^2$, and a spring constant in N/m is in kg/s^2 .

ⓘ Clock, ruler and reference mass

A balance compares weights at the same g , equal spring forces compare accelerations, and spring periods compare inertia. Each gives a ratio, and agreed standards turn ratios into seconds, metres and kilograms. All three SI units are now defined through constants of nature rather than by a particular object:

- **Time:** the SI second is defined by the frequency of a transition in the cesium-133 atom. Optical atomic clocks can achieve greater precision, but the SI definition uses cesium.
- **Length:** the metre is defined by fixing the speed of light in vacuum, together with the second.
- **Mass:** since 20 May 2019 the kilogram is defined by fixing the Planck constant. Kibble balances realize it by comparing a weight with an electromagnetic force measured electrically; calibrated physical reference weights are still used.

Sources: [SI definitions](#); [Atomic clocks](#); [The Kibble balance](#); [The 2018 SI redefinition](#); [Mass calibration](#).

⊙ Demonstration: measuring mass in orbit

Watch: [STEMonstrations, Momentum and Impulse \(NASA\)](#) ↗. In orbit a bathroom scale reads zero, so astronauts measure their mass with springs. On the International Space Station, ESA astronaut Andreas Mogensen uses a device in which two springs push with a known force while sensors measure the acceleration. On Skylab, in the 1970s, a spring-mounted chair measured mass from its period of oscillation, as in this section.

5 Damping of Spring Oscillations by Air Resistance

A real spring does not oscillate forever. Still air, for example, resists the block's motion through the room, and the oscillation dies away. Model this resistive force as proportional to the velocity and opposite to it,

$$F_x = -c\dot{x}, \quad c > 0.$$

Since $c\dot{x}$ is a force, c has units of $\text{N}/(\text{m/s}) = \text{kg/s}$.

The drag magnitude $c|\dot{x}|$ grows with speed and vanishes when the block is instantaneously at rest. Here the air is at rest; for air moving with velocity v_{air} , the linear model is $F_x = -c(\dot{x} - v_{\text{air}})$.

⚠ Drag from still air acts to slow the motion

Drag from still air always opposes the motion and acts to reduce the speed. Negative velocity means moving toward $-x$, not slowing down. A change from -5 m/s to -4 m/s increases the velocity but decreases the speed, from 5 to 4 m/s . For still air, $F_x = -c\dot{x}$

always opposes the motion: its contribution pushes the velocity toward zero from either side. The spring can still make the block speed up; drag opposes the motion.

Time between swings

Keep the same block, spring and air. As the swings get smaller, does the time between successive swings get longer, shorter, or stay the same?

The block now feels both the spring force, $-kx$, and air resistance, $-c\dot{x}$, which opposes its motion. Newton's second law therefore reads $m\ddot{x} = -kx - c\dot{x}$, or

$$m\ddot{x} + c\dot{x} + kx = 0.$$

This is the equation of damped motion. Without damping the solution was a cosine. With damping we expect an oscillation whose amplitude shrinks. A quantity that shrinks by the same factor in every equal time interval decays exponentially, as $e^{-\gamma t}$, where the decay rate $\gamma > 0$ has units of 1/s. So try an exponentially decaying cosine,

$$x(t) = Ae^{-\gamma t} \cos \omega t,$$

and find which γ and ω make it a solution. This trial function describes one particular motion: at $t = 0$ it has $x(0) = A$ and $\dot{x}(0) = -\gamma A$, with γ still to be found, so the block starts already moving.

$\therefore$ Deriving the damped motion

For a nonzero amplitude A , try

$$x(t) = Ae^{-\gamma t} \cos \omega t.$$

The cosine means the block still oscillates. Its amplitude is now time dependent: the envelope $|A|e^{-\gamma t}$ decreases as time passes.

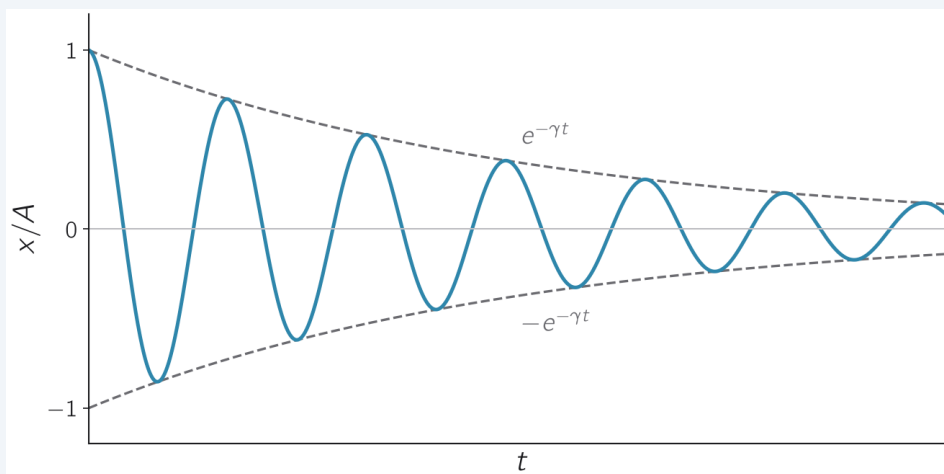


Figure 6: The cosine oscillates while $e^{-\gamma t}$ shrinks its amplitude. The graph shows x/A , with envelopes $\pm e^{-\gamma t}$. This zero-phase choice starts at $x = A$ with $\dot{x}(0) = -\gamma A$.

There are two time-dependent factors. Their derivatives are

$$\frac{d}{dt}e^{-\gamma t} = -\gamma e^{-\gamma t}, \quad \frac{d}{dt}\cos \omega t = -\omega \sin \omega t.$$

Apply the product rule, differentiating one factor at a time:

$$\begin{aligned}\dot{x} &= A \left[(-\gamma e^{-\gamma t}) \cos \omega t + e^{-\gamma t} (-\omega \sin \omega t) \right] \\ &= Ae^{-\gamma t} (-\gamma \cos \omega t - \omega \sin \omega t).\end{aligned}$$

For the second derivative, use the product rule again. The exponential gives one contribution; the bracket gives the other:

$$\begin{aligned}\ddot{x} &= A(-\gamma e^{-\gamma t}) (-\gamma \cos \omega t - \omega \sin \omega t) \\ &\quad + Ae^{-\gamma t} (\gamma \omega \sin \omega t - \omega^2 \cos \omega t) \\ &= Ae^{-\gamma t} [\gamma^2 \cos \omega t + \gamma \omega \sin \omega t + \gamma \omega \sin \omega t - \omega^2 \cos \omega t] \\ &= Ae^{-\gamma t} [(\gamma^2 - \omega^2) \cos \omega t + 2\gamma \omega \sin \omega t].\end{aligned}$$

The two equal sine terms are the origin of the factor of two.

Now substitute each expression into $m\ddot{x} + c\dot{x} + kx = 0$:

$$\begin{aligned}0 &= mAe^{-\gamma t} [(\gamma^2 - \omega^2) \cos \omega t + 2\gamma \omega \sin \omega t] \\ &\quad + cAe^{-\gamma t} [-\gamma \cos \omega t - \omega \sin \omega t] \\ &\quad + kAe^{-\gamma t} \cos \omega t.\end{aligned}$$

Divide by the common nonzero factor $Ae^{-\gamma t}$ and collect the cosine and sine terms separately:

$$0 = \underbrace{[m(\gamma^2 - \omega^2) - c\gamma + k]}_{\text{cosine coefficient}} \cos \omega t + \underbrace{[2m\gamma\omega - c\omega]}_{\text{sine coefficient}} \sin \omega t.$$

This equation must hold at every time. At $t = 0$, the cosine is one and the sine is zero, so the cosine coefficient must vanish. At $t = \pi/(2\omega)$, the cosine is zero and the sine is one, so the sine coefficient must vanish as well. We are looking for oscillations, with $\omega > 0$.

Start with the sine coefficient:

$$2m\gamma\omega - c\omega = 0 \implies \omega(2m\gamma - c) = 0 \implies \gamma = \frac{c}{2m}.$$

With this value, the two sine contributions cancel exactly. Then use $c = 2m\gamma$ in the cosine coefficient:

$$0 = m(\gamma^2 - \omega^2) - c\gamma + k = -m\gamma^2 - m\omega^2 + k.$$

Therefore

$$m\omega^2 = k - m\gamma^2 \implies \omega^2 = \frac{k}{m} - \gamma^2, \quad \omega = \sqrt{\frac{k}{m} - \gamma^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}.$$

A real nonzero ω requires $\gamma^2 < k/m$: this is the **underdamped** case, in which the block still oscillates. The units check: $\gamma = c/(2m)$ has units 1/s, and both k/m and γ^2 have units $1/s^2$. When $c \rightarrow 0$, the exponential stops decaying and $\omega \rightarrow \sqrt{k/m}$, recovering the undamped oscillator.

For underdamped motion, $c < 2\sqrt{mk}$, substituting the values of γ and ω gives

$$x(t) = Ae^{-\frac{c}{2m}t} \cos\left(\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} t\right).$$

The mass m , spring constant k , and air-resistance coefficient c determine how fast the swings decay and how long they take. For this chosen cosine, $A = x(0)$ is the initial displacement.

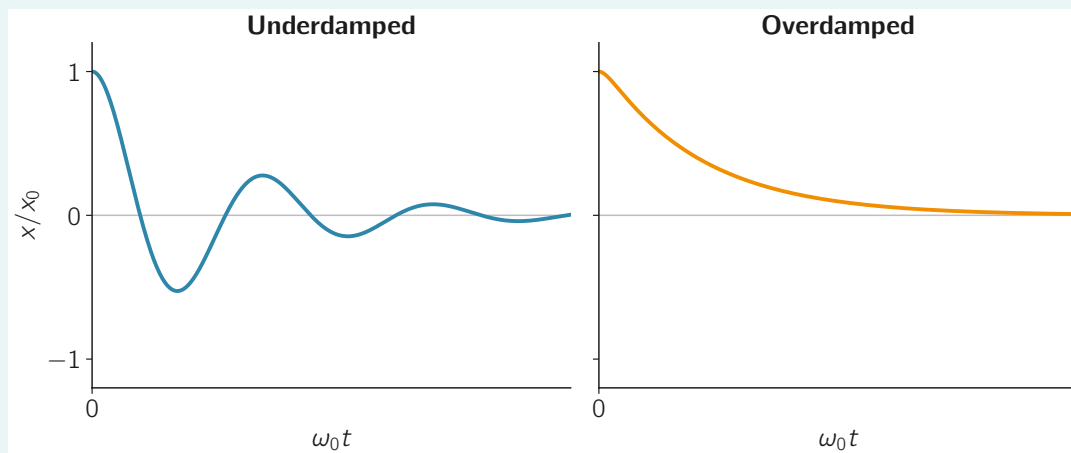
For this underdamped motion, the block still oscillates. For a given block, spring and surrounding air, m , k and c are fixed, so γ and ω are constant. The period $2\pi/\omega$ is longer than the period without air resistance, $2\pi\sqrt{m/k}$. Each oscillation takes the same time, even as the swings get smaller.

✓ Limiting cases: when does the spring stop oscillating?

Without air resistance ($c = 0$), the spring keeps oscillating with constant amplitude. For any $c > 0$, the motion eventually dies away as $t \rightarrow \infty$.

If air resistance is weak, $0 < c < 2\sqrt{mk}$, the block oscillates but the swings gradually die away: this is underdamped motion. If $c > 2\sqrt{mk}$, the drag is too strong for oscillations: this is overdamped motion. Released from rest away from equilibrium, the block returns without crossing it. The boundary $c = 2\sqrt{mk}$ is called critical damping.

More drag does not mean a faster return! For very large c , the return becomes slow, with a characteristic time approximately c/k .



Both blocks start at the same displacement x_0 and are released from rest. The plots use the same scales, with $\omega_0 = \sqrt{k/m}$.

⊙ Demonstration: damping

Damping in the Differential Equations Lab ↗. Select only $-kx$ and $-c\dot{x}$, then vary c . To reproduce the zero-phase cosine above, set $\dot{x}(0) = -cx(0)/(2m)$; starting from rest instead gives a phase-shifted oscillation in the underdamped regime.

6 Free-Body Diagrams and Friction

6.1 A block at rest on a table

To apply Newton's second law, draw the body and every force acting on it: a **free-body diagram**. Take a block of mass m resting on a horizontal table, Figure 7, and choose $+\hat{y}$ pointing up. Two forces act: gravity $\mathbf{F}_g = -mg\hat{y}$ and the normal force from the table $\mathbf{N} = N\hat{y}$. The block does not move, so its momentum is constant and $\ddot{y} = 0$.

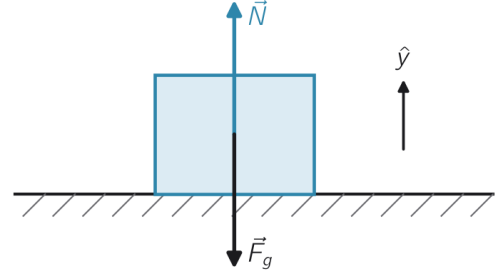


Figure 7: At rest on a table: $\mathbf{F}_g$ and $\mathbf{N}$, drawn from the center, cancel.

$$\frac{d\mathbf{p}}{dt} = 0 \implies \mathbf{F}_{\text{tot}} = \mathbf{F}_g + \mathbf{N} = 0 \implies N - mg = 0, \quad N = mg.$$

The net force vanishes because the momentum stays constant, not because it is zero at one instant. A ball thrown straight up has $\mathbf{p} = 0$ at the top of its flight, yet gravity still acts on it: there $d\mathbf{p}/dt = -mg\hat{y} \neq 0$.

The normal force is not always mg . It takes whatever value keeps the block from sinking into the table; here that value is mg because these are the only vertical forces and the vertical momentum does not change. On an incline, in [Section 7](#), it is different.

6.2 Static friction

Now a person pushes the block to the right with a small horizontal force $\mathbf{F}_p$, Figure 8. If the block still does not move, the net horizontal force is zero, so another horizontal force must cancel the push. This is **static friction**, exerted by the table on the block along their contact surface:

$$\mathbf{f}_s = -\mathbf{F}_p.$$

While the block remains at rest, static friction balances the applied horizontal force, whatever its size, but only up to a maximum.

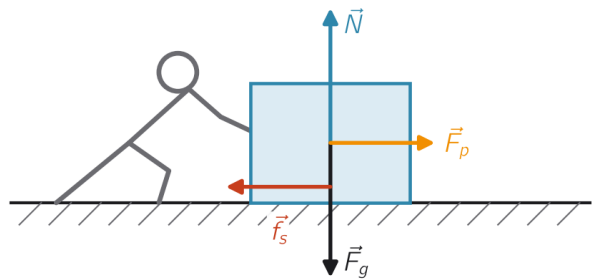


Figure 8: A person pushes the block with $\mathbf{F}_p$ and it stays at rest. Static friction $\mathbf{f}_s$, along the contact surface, balances the push; $\mathbf{N}$ balances $\mathbf{F}_g$. Equal arrow lengths mark the balanced pairs.

The push is itself a contact force. The hand presses perpendicular to the block's face, so $\mathbf{F}_p$ is the normal force of the hand on the block, and the block presses back on the hand. Friction is

the part of a contact force that lies along the surface; the normal force is the part perpendicular to it.

That maximum comes from the block and the table pressing on each other, so it grows with how hard they press, measured by the normal force N . Static friction can take any magnitude up to the maximum:

$$|\mathbf{f}_s| \leq \mu_s N.$$

The dimensionless number μ_s is the phenomenological **coefficient of static friction**; it depends on the pair of surfaces and their condition. On the table the actual static friction equals the push and reaches $\mu_s N$ only at the threshold of sliding, $F_p = \mu_s N$. There static friction is at its largest value and still balances the push; a push even slightly larger cannot be balanced, and the block starts to slide.

For a numerical example, take $m = 2 \text{ kg}$, $g \approx 10 \text{ m/s}^2$ and the illustrative value $\mu_s \approx 0.5$. Then $N = mg \approx 20 \text{ N}$ and the largest push the block can resist is

$$F_{p,\text{max}} = \mu_s N \approx 0.5 \times 20 \text{ N} = 10 \text{ N}.$$

For comparison, 10 N is about the weight of a 1 kg mass. Lifting the same 2 kg block steadily would require about 20 N, independent of the height through which it is lifted. Real coefficients vary with the materials and their surfaces; 0.5 is a number for the estimate, not a property of wood or of tables.

⊙ Demonstration: the normal force under an angled pull

Pull at an Angle ↗. Tilt the rope upward: the normal force drops below mg , and so does $\mu_s N$.

6.3 Kinetic friction

Once the surfaces slide, friction is **kinetic**, Figure 9, with magnitude

$$|\mathbf{f}_k| = \mu_k N,$$

directed opposite to the relative sliding velocity at the contact. In this model $|\mathbf{f}_k|$ does not depend on the sliding speed; the speed-dependent resistive force is the drag $-c\dot{x}$ of Section 5. The dimensionless μ_k is the **coefficient of kinetic friction**. Usually $\mu_s \geq \mu_k$: the kinetic friction during sliding is no larger than the maximum static friction that had to be overcome to start it.

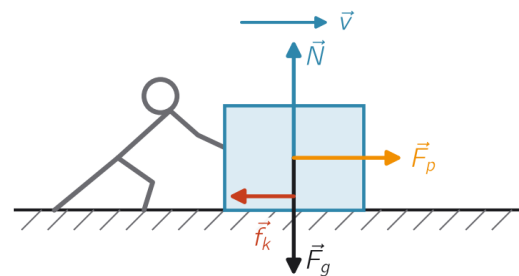


Figure 9: Sliding with velocity $\mathbf{v}$: kinetic friction $\mathbf{f}_k$ opposes the sliding and is smaller than the push here.

Before the block slides, the actual static friction equals the push, so a gentle push meets a static friction smaller than $\mu_k N$. While the block slides under the push, the net horizontal force is $F_p - \mu_k N$.

A moving body can keep its momentum constant. If the block slides and the push is adjusted to exactly $F_p = \mu_k N$, the horizontal forces balance and the block slides with constant momentum.

① Microscopic origin of friction

Why does friction depend on the normal force? Even smooth-looking surfaces touch only at small patches. Pressing harder deforms these contacts and increases their total area. Sliding requires shearing the contacts as they break and reform: more real contact means more resistance to sliding.

The real contact area is a small fraction of the apparent area of the touching faces, and it is the real area that grows with N . A brick lying on its broad face and the same brick standing on its end press on the floor with the same N . The broad face spreads the load over more patches, each pressed less, and the real contact area comes out about the same. Within this model, friction does not depend on the apparent area of contact.

A simple model makes the connection. If the average pressure at the contacts and the force needed to shear each unit of contact area are roughly constant, both N and $|f_k|$ are proportional to the real contact area. Their ratio is then approximately constant:

$$\frac{|f_k|}{N} \approx \mu_k.$$

For soft metals, local yielding can keep the contact pressure roughly fixed. This is one microscopic explanation of the approximate law, not a universal rule for every surface.

So mass can matter: on a horizontal table, $N = mg$, and a heavier block generally experiences more friction. It is the coefficient that we treat as fixed for the same surfaces and conditions. The normal force is the total load; pressure is force per area.

Contacts can also grow or strengthen while at rest. Sliding continually renews them, helping explain why maximum static friction often exceeds kinetic friction.

Further reading: [Feynman on friction](#).

Some approximate coefficients. Both coefficients are dimensionless. These are illustrative textbook values, not universal constants for a material pair; surface finish, contamination, moisture and temperature can change them substantially.

Surfaces in contact	μ_s	μ_k
Wood on wood	0.5	0.3
Rubber on dry concrete	1.0	0.7
Rubber on wet concrete	0.5–0.7	0.3–0.5
Steel on steel, dry	0.6	0.3
Steel on steel, oiled	0.05	0.03
Teflon on steel	0.04	0.04
Ice on ice	0.1	0.03

Source: [OpenStax, approximate friction coefficients \(Table 6.1\)](#).

⊙ **Demonstration: static and kinetic friction**

Static and Kinetic Friction on a Crate ↗. Increase the push and watch friction match it up to the static threshold, then change once sliding begins. Vary μ_s and μ_k to compare the two regimes.

7 The Inclined Plane

Place the block on a fixed plane tilted at angle α to the horizontal. The block slides without rotating and is treated as a point mass. The only forces on it are gravity and the forces from the plane: the normal force and, once we add it, friction.

7.1 Axes along and perpendicular to the plane

The block cannot go through the plane, so while they are in contact the plane pushes on it perpendicular to the surface. Choose unit vectors adapted to the plane, Figure 10(a): $\hat{n}$ perpendicular to the surface, pointing outward, and $\hat{t}$ along the surface, pointing downhill. In these axes the normal force is

$$\mathbf{N} = N\hat{n}.$$

Gravity is $\mathbf{F}_g = -mg\hat{y}$. The vertical $\hat{y}$ makes angle α with $\hat{n}$, so $\hat{y} \cdot \hat{n} = \cos \alpha$, and $\hat{y}$ has a component $-\sin \alpha$ along the downhill $\hat{t}$. Thus $\hat{y} = \cos \alpha \hat{n} - \sin \alpha \hat{t}$ and

$$\mathbf{F}_g = -mg \cos \alpha \hat{n} + mg \sin \alpha \hat{t},$$

as in Figure 10(b). The downhill component is positive because $\hat{t}$ points downhill and gravity pulls the block down the slope.

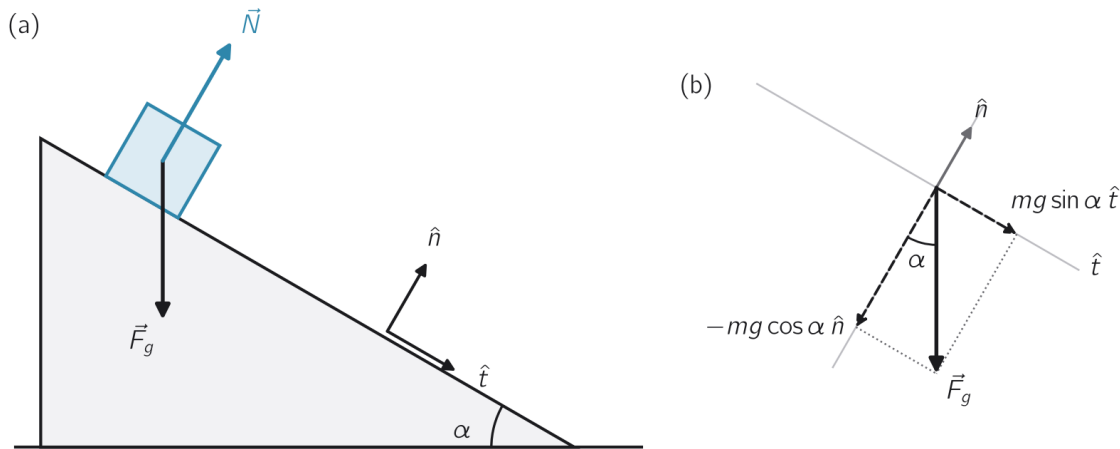


Figure 10: (a) A block on a frictionless incline at angle α , with outward normal $\hat{n}$ and downhill direction $\hat{t}$. $\mathbf{N}$ and $\mathbf{F}_g$ are drawn from the block's center. (b) Gravity resolved along the plane's axes: $-mg \cos \alpha \hat{n}$ and $mg \sin \alpha \hat{t}$. The angle between $\mathbf{F}_g$ and $-\hat{n}$ equals the incline angle α . Drawn to one scale within each panel, for $\alpha = 30^\circ$.

7.2 The frictionless incline

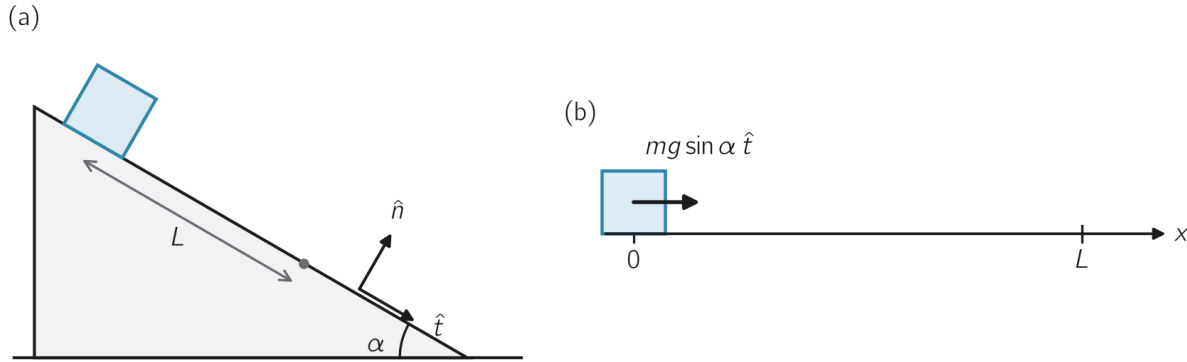


Figure 11: (a) The block starts at rest and slides a distance L down the slope. (b) The motion along the slope is one-dimensional: x is measured downhill from 0 to L , and the only force along it is $mg \sin \alpha \hat{t}$.

The block stays on the surface, so its momentum along $\hat{n}$ stays zero and does not change:

$$\mathbf{F}_{\text{tot}} \cdot \hat{n} = 0 \implies \mathbf{F}_g \cdot \hat{n} + N = 0 \implies -mg \cos \alpha + N = 0, \quad N = mg \cos \alpha.$$

On the incline the normal force is smaller than mg . Without friction, the only force along the slope is the downhill component of gravity:

$$\mathbf{F}_{\text{tot}} \cdot \hat{t} = mg \sin \alpha, \quad \mathbf{F}_{\text{tot}} = mg \sin \alpha \hat{t}.$$

Let x be the distance traveled down the slope from the starting point, Figure 11. Newton's second law along $\hat{t}$ gives

$$m\ddot{x} = mg \sin \alpha \implies \ddot{x} = g \sin \alpha.$$

This is motion with constant acceleration. Read the derivatives backwards: $\dot{x} = g \sin \alpha t + v_0$, and starting from rest $v_0 = 0$; then $x = \frac{1}{2}g \sin \alpha t^2$, with $x(0) = 0$. The block covers a length L of slope when $L = \frac{1}{2}g \sin \alpha t^2$, that is, after

$$t_L = \sqrt{\frac{2L}{g \sin \alpha}}.$$

The time does not depend on the mass.

✓ Limits of the frictionless incline

As $\alpha \rightarrow 90^\circ$, the normal force tends to zero and the motion approaches free fall, $L = \frac{1}{2}gt^2$. As $\alpha \rightarrow 0$, the downhill acceleration tends to zero and the time to travel L grows without bound.

7.3 Static friction on the incline

✍ Angle at which sliding starts

A plane is tilted slowly until a block resting on it starts to slide. Compare a heavier and a lighter block whose surfaces have the same coefficient of static friction with the plane. Does the angle at which sliding starts change? Does it change if the experiment is done on the Moon?

Now add friction. If the block is at rest, the net force along the slope is zero, so static friction must point uphill and cancel the downhill component of gravity, Figure 12(a):

$$\vec{f}_s = -mg \sin \alpha \hat{t}.$$

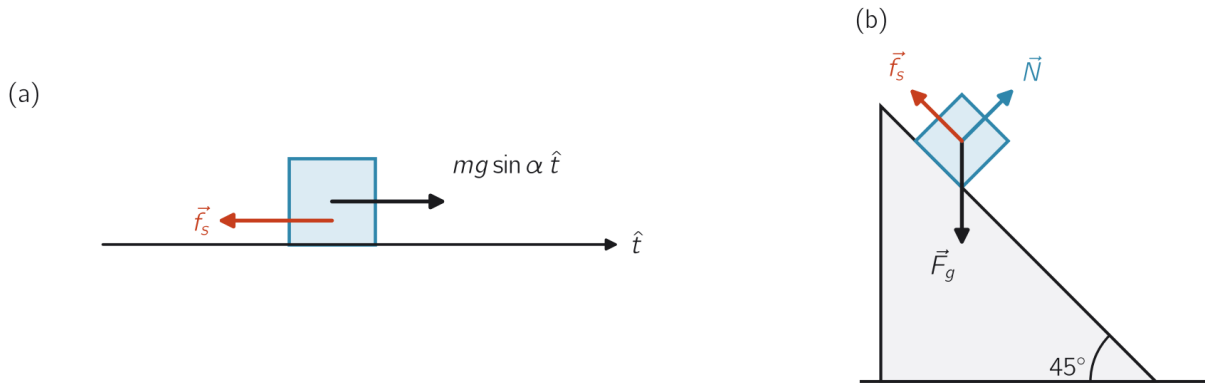


Figure 12: (a) Along the slope, a block at rest: static friction $\vec{f}_s$ cancels $mg \sin \alpha \hat{t}$, for any angle with $\tan \alpha \leq \mu_s$. (b) At $\alpha_c = 45^\circ$ with $\mu_s = 1$, static friction reaches its maximum, $\mu_s N = mg \sin 45^\circ$. Forces are drawn from the block's center, to one scale within each panel.

It can do so only up to its maximum, $\mu_s N$, and we already know $N = mg \cos \alpha$. The block stays at rest as long as

$$mg \sin \alpha \leq \mu_s mg \cos \alpha \quad \Longleftrightarrow \quad \tan \alpha \leq \mu_s.$$

Both m and g have cancelled. You could do this experiment on the Moon: provided the surfaces have the same coefficient μ_s , the critical angle would be the same as on Earth. Both the downhill pull and the maximum static friction decrease in proportion to g , so their ratio is unchanged. At the critical angle static friction is at its maximum and still holds the block; at any steeper angle the block slides. The critical angle is

$$\alpha_c = \arctan \mu_s.$$

This also gives a simple way to measure the coefficient of static friction: slowly tilt the plane until the block just begins to slide, measure the angle at the onset of sliding, and calculate $\mu_s = \tan \alpha_c$. Within this model, you do not need to know the block's mass or the local value of g . Two wooden blocks of very different mass on the same wooden plane start to slide at the

same angle. A metal block on that plane has a different μ_s and starts to slide at a different angle.

For example, $\mu_s = 1$ gives $\alpha_c = \arctan 1 = 45^\circ$, where $\sin 45^\circ = \cos 45^\circ = \sqrt{2}/2$, Figure 12(b).

7.4 Sliding down the incline

Return to $\alpha = 45^\circ$ with $\mu_s = 1$, where static friction holds the block at its maximum, and take $\mu_k = 0.5$. After a small nudge the block slides down the slope, and kinetic friction acts. It opposes the sliding, so it points uphill, along $-\hat{t}$, with magnitude $\mu_k N = \mu_k mg \cos \alpha$. At 45° this is half the downhill component of gravity, so the forces no longer balance and the block accelerates down the slope, Figure 13.

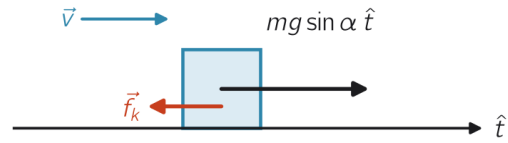


Figure 13: At 45° with $\mu_k = 0.5$, $\mathbf{f}_k$ is half the downhill component; the net force points down the slope.

The net force along the slope is

$$\mathbf{F}_{\text{tot}} \cdot \hat{t} = mg \sin \alpha - \mu_k mg \cos \alpha = mg \frac{\sqrt{2}}{2} - 0.5 mg \frac{\sqrt{2}}{2} = mg \frac{\sqrt{2}}{2} (1 - 0.5) = mg \frac{\sqrt{2}}{4}.$$

The friction term enters with a minus sign because friction opposes the downhill motion. This is the total force along the slope, so the acceleration down the slope is

$$\ddot{x} = g \frac{\sqrt{2}}{4} \approx \frac{10 \text{ m/s}^2 \times 1.41}{4} \approx 3.5 \text{ m/s}^2.$$

In general, a block sliding *down* has $\ddot{x} = g(\sin \alpha - \mu_k \cos \alpha)$. A block sliding *up* the slope would feel kinetic friction pointing downhill instead, and this formula would not apply.

7.5 Friction as the plane is tilted

Now slowly increase the angle. How does the friction change? Call the downhill driving force $F = mg \sin \alpha$. While the block stays at rest, friction matches it: $|\mathbf{f}_s| = F$. This gives the straight rising part of Figure 14, up to $F_c = mg \sin \alpha_c$, where static friction reaches its maximum.

Once the block slides downhill, $|\mathbf{f}_k| = \mu_k mg \cos \alpha$. For $\mu_k < \mu_s$, the friction drops when sliding begins. It then decreases as we increase the angle further: the plane is pushing less strongly on the block. To plot this against F , write the cosine in terms of the sine, $\cos \alpha = \sqrt{1 - \sin^2 \alpha}$ for $0 \leq \alpha \leq 90^\circ$, and use $\sin \alpha = F/(mg)$:

$$|\mathbf{f}_k| = \mu_k mg \cos \alpha = \mu_k mg \sqrt{1 - \sin^2 \alpha} = \mu_k mg \sqrt{1 - \left(\frac{F}{mg}\right)^2}.$$

At $\alpha = 90^\circ$, $F = mg$ and the normal force, and hence friction, vanish. Unlike the horizontal pushing experiment, tilting changes the normal force too.

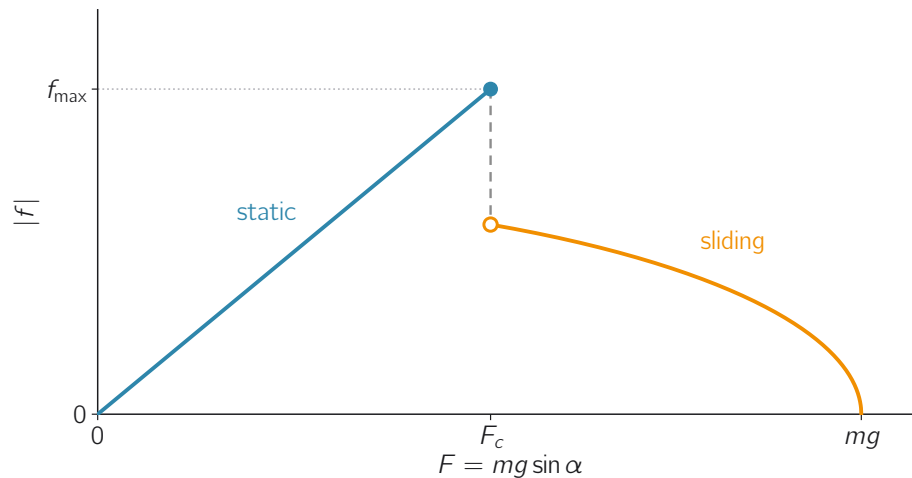


Figure 14: Friction as the incline angle increases, starting with the block at rest. The filled point is the largest static friction; the open point marks the kinetic value just after sliding begins. The dashed drop connects the two regimes.

△ Friction at a vertical wall

At $\alpha = 90^\circ$, $N = 0$, so the friction law allows no friction and the block falls. Yet a sticky note stays on a wall. The note sticks to the wall, an effect called adhesion, and $|\mathbf{f}| \leq \mu_s N$ leaves it out. The friction law is a simple model of a complicated contact.

⊙ Demonstration: tilting a rough ramp

Tilting a Rough Ramp ↗. Tilt a wooden ramp carrying a light and a heavy wooden block, and compare the angles at which they start to slide. The graph traces the friction of Figure 14.

8 Walking

What pushes you forward when you start walking? Your velocity changes, so the net horizontal force on you is not zero. Your foot pushes backward on the floor. If it does not slip, static friction from the floor pushes forward on you. The backward force acts on the floor, not on you. On a frictionless floor the foot would slide backward and nothing would push you forward: you could not start walking.

To draw a **free-body diagram**, isolate the person and keep only the forces acting on that person. During the push-off shown in Figure 15, these are gravity, the normal forces at the feet, and forward static friction. If the vertical momentum does not change, $N_1 + N_2 = mg$; horizontally, $dp_x/dt = f_s$. Friction opposes the foot's tendency to slip backward against the floor, so here it helps the person speed up. Here we are looking at the push-off part of a step.

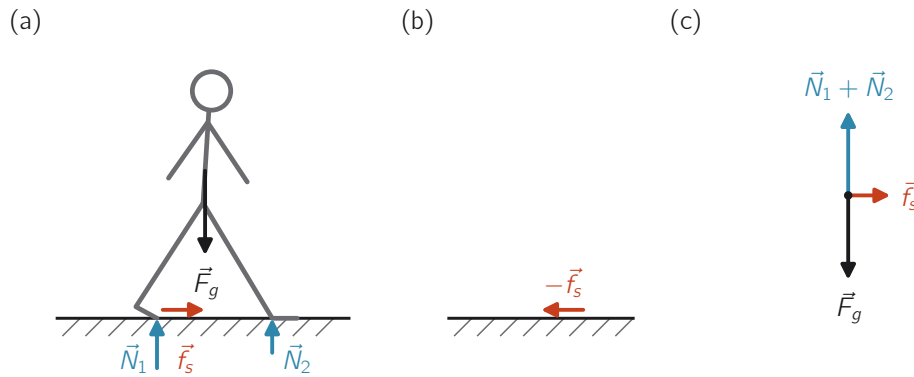


Figure 15: Starting to walk. (a) Forces on the person during push-off. (b) The backward push acts on the floor. (c) Only forces on the person belong in the person's free-body diagram.

⦿ Demonstration: walking without friction

Watch: Animation vs. Physics (Alan Becker) ↗. Early in the video the stick figure walks onto a frictionless floor and can no longer move himself forward by walking.

9 String Tension and Pulleys

A taut string pulls along its length. The magnitude of this pull is the **tension**, T . We model the string as massless and inextensible, and the fixed pulley as massless with a frictionless axle. These assumptions do different jobs: negligible mass gives equal tension along the string, while fixed length links the motions of its ends.

Imagine cutting a short straight piece out of the string. Replace the missing string on each side by the force it exerts on the piece. These two tensions point in opposite directions. Newton's second law gives

$$T_2 - T_1 = m_{\text{piece}}a.$$

For a massless piece with finite acceleration, $T_2 = T_1$. We can make this cut anywhere along a straight, unsupported part of the string: the tension has the same magnitude throughout.

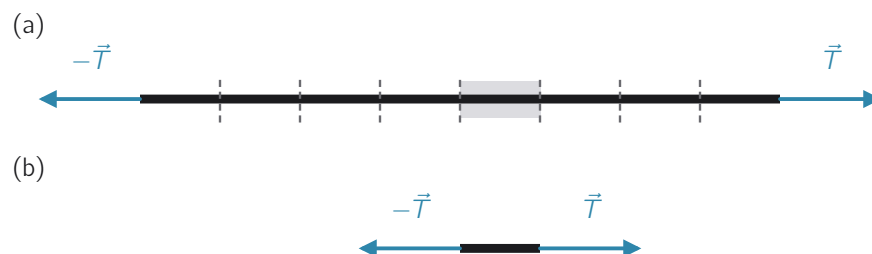


Figure 16: (a) Imagine cutting a piece out of a taut string. (b) The neighboring pieces pull on the isolated piece from opposite ends. For a massless piece, the tensions balance.

What happens where the string bends around the pulley? The two end tensions now point in different directions, so they do not cancel each other as vectors. The pulley pushes on the string too. For a massless piece, its contact force balances the vector sum of the tensions. On a frictionless pulley there is no tangential contact force, so the tension's magnitude stays the same while its direction changes.

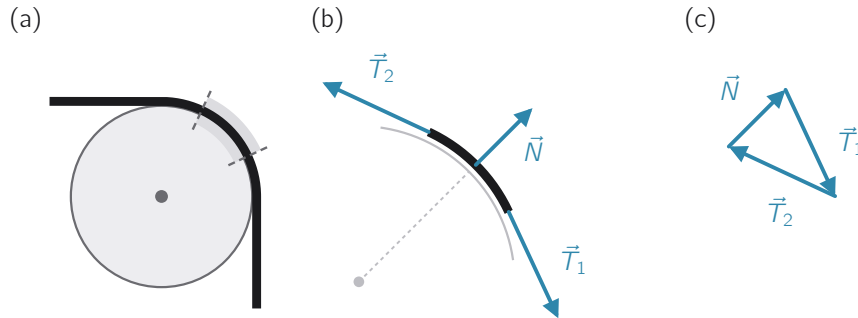


Figure 17: (a) String bending around a pulley. (b) The isolated piece feels two tensions and a resultant contact force N from the pulley. (c) Together these force vectors close a triangle. The bend is exaggerated for visibility.

Keeping the string's length fixed is a separate condition. Around a fixed pulley, one end takes up exactly the length released by the other, so the ends have the same *speed along the string*. Their velocity vectors need not be equal: they point along different parts of the string. We should not call this zero relative velocity.

9.1 A hanging mass and a block on an incline

Connect a mass M hanging vertically to a block of mass m on a frictionless incline, Figure 18. Draw a separate free-body diagram for each block. The hanging block feels T upward and Mg downward. The other block feels gravity, the normal force, and tension up the slope; perpendicular to the slope, $N = mg \cos \alpha$.

First ask when the system can stay at rest. Each block must have zero net force:

$$T = Mg, \quad T = mg \sin \alpha \quad \implies \quad \sin \alpha = \frac{M}{m}.$$

The balance angle depends only on the ratio of the two masses.

What if the forces do not balance? Choose positive motion down the slope for m and upward for M . Because the string keeps its length, both coordinates have the same acceleration a . Newton's second law gives

$$mg \sin \alpha - T = ma, \quad T - Mg = Ma.$$

Add the equations to eliminate the unknown tension:

$$a = g \frac{m \sin \alpha - M}{m + M}, \quad T = M(g + a).$$

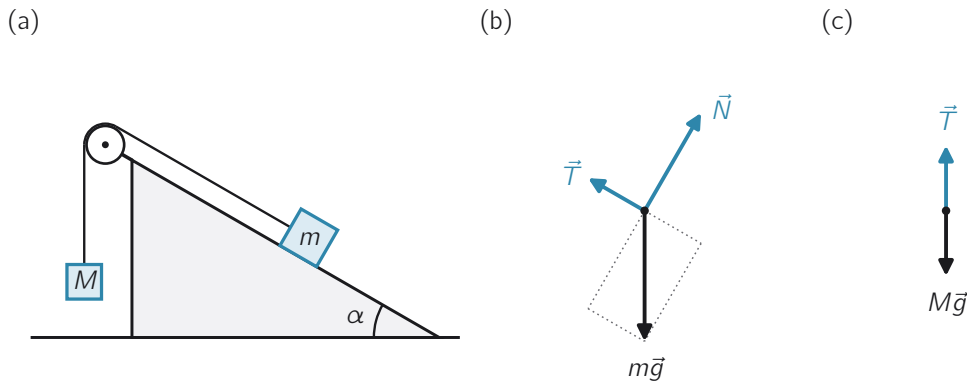


Figure 18: (a) A hanging mass connected to a block on a frictionless incline. (b,c) Separate free-body diagrams for the two blocks, shown in equilibrium. The same tension pulls along both ends of the ideal string; fixed length links their accelerations.

If $m \sin \alpha > M$, the block accelerates down the slope and lifts M . If $m \sin \alpha < M$, a is negative: the acceleration is opposite to our chosen directions. At equality we recover the balanced case. The tension is fixed by the motion; it is not automatically equal to either block's weight.

✓ Limits of the incline and hanging mass

Start with the balance angle, $\sin \alpha = M/m$. As $M \rightarrow 0$, $\alpha \rightarrow 0$: a very light hanging mass holds the block only on a nearly flat plane. At $M = m$, $\alpha = 90^\circ$: two equal masses hang side by side over the pulley. For $M > m$, $\sin \alpha = M/m > 1$ has no solution; on a frictionless plane no angle holds the system at rest, and M descends.

The acceleration $a = g(m \sin \alpha - M)/(m + M)$ is g times a ratio of masses, so it has units of m/s^2 . With $M = 0$ it is $g \sin \alpha$, the frictionless incline of [Section 7](#). As $m \rightarrow 0$ it tends to $-g$: M falls freely. At $\alpha = 90^\circ$ it is $g(m - M)/(m + M)$, two masses hanging over one pulley.

⊙ Demonstration: incline and hanging mass

Incline and Hanging Mass ↗. Change α and M/m and watch which way the blocks move; the tension is drawn beside Mg and $mg \sin \alpha$.

10 A Car on a Circular Track

A car follows a circle of radius R on a flat road at constant speed v . What keeps it turning? Draw the forces first: gravity downward, the normal force upward, and static friction from the road through the tires. If we include air resistance, it acts backward. The tires roll without sliding.

Look from above as well as from the side, [Figure 19](#). Choose $\hat{n}$ vertically upward, $\hat{r}$ horizontally outward from the center, and $\hat{\theta}$ along the motion. The car stays at the same height and its speed is constant. Take the dot product of the net force with each unit vector to find its

component in that direction. Here $\hat{\mathbf{n}}$ is normal to the horizontal plane of the orbit. The normal and tangential components vanish; the radial component points inward:

$$\begin{aligned}\mathbf{F}_{\text{net}} \cdot \hat{\mathbf{n}} &= N - mg = 0, \\ \mathbf{F}_{\text{net}} \cdot \hat{\boldsymbol{\theta}} &= f_{\theta} - D = 0, \\ \mathbf{F}_{\text{net}} \cdot \hat{\mathbf{r}} &= f_r = m(\ddot{r} - r\dot{\theta}^2).\end{aligned}$$

Here D is the magnitude of air resistance. Look at the radial component from polar coordinates. On a circle, $r = R$ is constant, so $\ddot{r} = 0$. The tangential velocity is $v_{\theta} = r\dot{\theta}$. Since $\dot{r} = 0$, this is the whole speed: $v = R\omega$, where $\omega = \dot{\theta}$. Thus

$$a_r = \ddot{r} - r\dot{\theta}^2 = 0 - R\omega^2 = -R\left(\frac{v}{R}\right)^2 = -\frac{v^2}{R}.$$

Therefore $\mathbf{F}_{\text{net}} \cdot \hat{\mathbf{r}} = f_r = -mv^2/R$. The minus sign says that the force points inward: the radial part of the road's friction is $f_r \hat{\mathbf{r}} = -(mv^2/R) \hat{\mathbf{r}}$, of magnitude mv^2/R , toward the center. It changes the direction of the velocity, even though the speed stays constant.

The forward driving force is the road's static friction: $f_{\theta} = D$. The engine turns the wheels, the tires push backward on the road, and the road pushes forward on the tires. There is no separate engine force to add to the free-body diagram of the whole car. The tires' backward push acts on the road, and through it on the Earth; the Earth's mass is so large that the resulting change in its velocity is negligible.

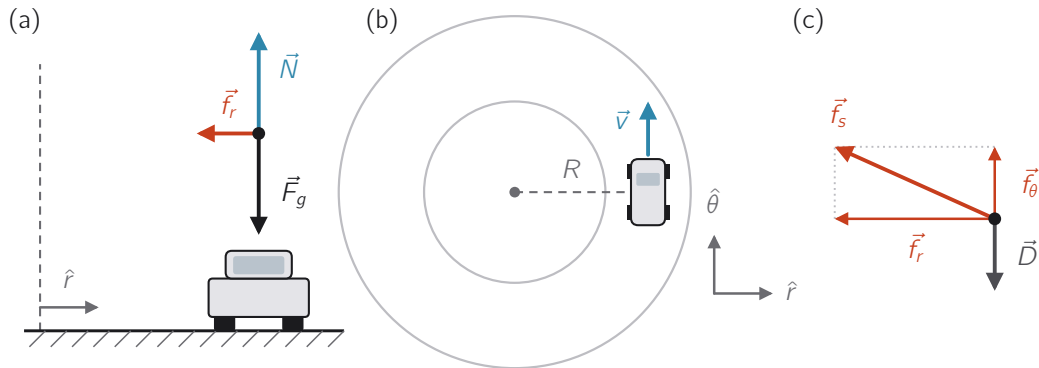


Figure 19: A powered car turning at constant speed. (a) The radial-vertical view shows gravity, the normal force, and the inward component of friction. (b) From above, the car moves tangentially while turning inward. (c) Road friction supplies both turning and forward traction; air resistance D balances the forward component.

First take the ideal limit with no air resistance. At constant speed, $f_{\theta} = 0$, so the required friction is just mv^2/R . It must stay within the available static friction:

$$\frac{mv^2}{R} \leq \mu_s mg \quad \implies \quad \mu_s \geq \frac{v^2}{Rg}.$$

For $v = 30 \text{ m/s}$, about 67 mph, and $R = 100 \text{ m}$, taking $g \simeq 10 \text{ m/s}^2$,

$$\frac{v^2}{Rg} = \frac{(30 \text{ m/s})^2}{100 \text{ m} \times 10 \text{ m/s}^2} = 0.9,$$

so the turn needs $\mu_s \geq 0.9$. The mass cancels. Static friction is not always $\mu_s mg$; that is its maximum.

Will the car slip? For illustration, take $\mu_s = 1.0$ on dry asphalt and $\mu_s = 0.5$ on wet asphalt. In this ideal comparison, the dry road supplies enough friction for the turn; the wet road does not. The car then slips outward relative to the intended circular path. These are illustrative coefficients: the actual values depend on the tires and road conditions. Water generally reduces the available grip ([FHWA: wet-pavement friction](#)).

With air resistance, or when speeding up or braking, the limit applies to the *total* road friction:

$$\sqrt{f_r^2 + f_\theta^2} \leq \mu_s mg.$$

Some of the available friction is then needed along the track, leaving less for turning.

10.1 A banked road

What if we tilt the road? The normal force now has an inward component, so friction does not have to provide the whole turn. Let the road rise outward at an angle α to the horizontal. Keep the same directions: $\hat{n}$ is vertical, $\hat{\theta}$ follows the motion, and $\hat{r}$ points outward. The road's normal force $\vec{N}$ is perpendicular to the tilted surface, not parallel to $\hat{n}$.

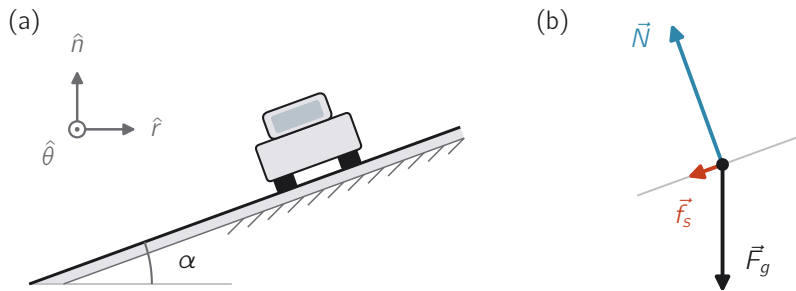


Figure 20: A banked turn. (a) The road rises outward through an angle α ; $\hat{n}$ remains vertical. (b) For a car tending to slip up the bank, static friction points down the slope. The normal force and friction together provide the inward pull.

Draw gravity, the normal force, and friction, Figure 20. Consider a car tending to slip up the bank: static friction points down the slope. Call its magnitude f_s . Neglect air resistance and keep the speed constant. Resolve the forces in the same order:

$$\vec{F}_{\text{net}} \cdot \hat{n} = N \cos \alpha - f_s \sin \alpha - mg = 0,$$

$$\vec{F}_{\text{net}} \cdot \hat{\theta} = 0,$$

$$\vec{F}_{\text{net}} \cdot \hat{r} = -N \sin \alpha - f_s \cos \alpha = -\frac{mv^2}{R}.$$

The vertical forces balance. The tangential forces balance. The radial acceleration is the same as on the flat road: $\ddot{r} - r\dot{\theta}^2 = -R\omega^2 = -v^2/R$. Banking changes which forces provide the inward pull.

As long as the tires do not slip, $f_s \leq \mu_s N$. At the threshold of slipping up the bank, static friction reaches its maximum: $f_s = \mu_s N$. This equality describes the threshold, not every turn without slipping. Divide the inward-force equation by the vertical one to eliminate N and m :

$$\frac{\sin \alpha + \mu_s \cos \alpha}{\cos \alpha - \mu_s \sin \alpha} = \frac{v^2}{Rg}.$$

∴ Finding the bank angle

Divide the numerator and denominator on the left by $\cos \alpha$:

$$\frac{\tan \alpha + \mu_s}{1 - \mu_s \tan \alpha} = \frac{v^2}{Rg}.$$

Multiply by the denominator, then collect the terms containing $\tan \alpha$:

$$\begin{aligned}\tan \alpha + \mu_s &= \frac{v^2}{Rg} (1 - \mu_s \tan \alpha), \\ \tan \alpha + \mu_s \frac{v^2}{Rg} \tan \alpha &= \frac{v^2}{Rg} - \mu_s, \\ \left(1 + \mu_s \frac{v^2}{Rg}\right) \tan \alpha &= \frac{v^2}{Rg} - \mu_s.\end{aligned}$$

So the bank angle at this slipping threshold satisfies

$$\tan \alpha = \frac{v^2/(Rg) - \mu_s}{1 + \mu_s v^2/(Rg)}.$$

We used the maximum static friction only at the slipping threshold. At lower speeds friction can instead point up the slope, preventing the car from slipping down.

How much does banking help? Solve the same threshold equation for the maximum speed:

$$v_{\max} = \sqrt{Rg \frac{\tan \alpha + \mu_s}{1 - \mu_s \tan \alpha}}, \quad \mu_s \tan \alpha < 1.$$

For the same $R = 100$ m turn, take $g \approx 10$ m/s² and an illustrative $\mu_s = 0.9$. Keeping these fixed gives:

Bank angle α	$v_{\max}$ (miles per hour)
0°	67
5°	73
10°	80
15°	88

The tilted normal force supplies part of the inward pull, so the same tires can hold the turn at a higher speed in this model.

Now set $\mu_s = 0$. Even without friction, the bank can support a circular turn at the speed satisfying

$$\tan \alpha = \frac{v^2}{Rg}.$$

The car can keep going around the circle even without friction, provided its speed matches the bank angle: $v = \sqrt{Rg \tan \alpha}$. The vertical component of the normal force balances gravity; its horizontal component keeps turning the velocity inward. Banking lets the normal force do what friction alone had to do on the flat road.

☉ **Demonstration: a banked curve without friction**

The Icy Banked Curve ↗. A car on a frictionless bank holds its circle at only one speed. Enter slower or faster and watch which way it leaves the road.

These problems all follow the same program: choose the body, draw its forces, write Newton's second law in components, and impose the constraints and initial conditions. In principle, this lets us solve any problem within this classical model once the forces are specified. The equations may be complicated enough to need a numerical solution, but the starting point is the same.

Summary

- **Recap.** $\mathbf{F}_{\text{net}} = d\mathbf{p}/dt$, $\mathbf{p} = m\mathbf{v}$. A central force keeps $\mathbf{r} \times \mathbf{v}$ constant, so the areal rate $\frac{1}{2}|\mathbf{r} \times \mathbf{v}|$ is constant.
- **Two roles for mass.** $\mathbf{a} = -(m_g/m_i)GM_g\hat{\mathbf{r}}/r^2$. The weak equivalence principle and a normalization give $m_g = m_i$, so free fall cannot measure mass. Near Earth's surface, $\mathbf{F} \approx -mg\hat{\mathbf{r}}$, $g = GM_E/R_E^2$.
- **Forces.** Gravity and the electric force are fundamental; spring, normal, tension and friction forces are effective.
- **Spring.** $m\ddot{x} = -kx$, $T = 2\pi\sqrt{m/k}$; on the same spring, $m = m_{\text{ref}}(T/T_{\text{ref}})^2$. Dimensions T , L , M ; SI units s, m, kg.
- **Damping.** $m\ddot{x} + c\dot{x} + kx = 0$: underdamped solutions $x = e^{-\gamma t}(A \cos \omega t + B \sin \omega t)$ with constant $\gamma = c/(2m)$ and $\omega^2 = k/m - \gamma^2$; release from rest at x_0 gives $A = x_0$, $B = \gamma x_0/\omega$.
- **Friction.** Without relative sliding, $|\mathbf{f}_s| \leq \mu_s N$; during relative sliding, $|\mathbf{f}_k| = \mu_k N$, opposite the relative sliding and independent of its speed; usually $\mu_s \geq \mu_k$. The law does not depend on the apparent contact area and leaves out adhesion.
- **Incline.** $N = mg \cos \alpha$. Frictionless: $\ddot{x} = g \sin \alpha$, $t_L = \sqrt{2L/(g \sin \alpha)}$. At rest if $\tan \alpha \leq \mu_s$, so $\alpha_c = \arctan \mu_s$. Sliding down: $\ddot{x} = g(\sin \alpha - \mu_k \cos \alpha)$, which is $g\sqrt{2}/4 \approx 3.5 \text{ m/s}^2$ for $\alpha = 45^\circ$, $\mu_k = 0.5$.
- **Walking.** During push-off, static friction from the floor pushes the person forward; the backward reaction acts on the floor.
- **Ideal string.** Negligible mass gives equal tension; fixed length links the speeds along the string. At a pulley, include its contact force when balancing forces on a bent piece.
- **Incline and hanging mass.** At rest when $\sin \alpha = M/m$; otherwise $a = g(m \sin \alpha - M)/(m + M)$ down the slope, with $T = M(g + a)$.
- **Turning on a flat road.** Static friction supplies the inward force mv^2/R ; the car holds the turn if $\mu_s \geq v^2/(Rg)$, independent of its mass. A frictionless bank holds it at $\tan \alpha = v^2/(Rg)$.

Every body this week was pushed by something outside it: the Earth, the floor, a rope, the road. Two skaters who push off each other, or a cart that fires a ball, have only each other to push on. Each body's motion changes. What stays the same for the pair?